THE TEACHING OF MATHEMATICS IN SECONDARY SCHOOLS

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THE
TEACHING OF MATHEMATICS
IN
SECONDARY SCHOOLS

BY

ARTHUR SCHULTZE

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PREFACE

SINCE 1906 the author has delivered a yearly course of lectures at New York University on the Teaching of Mathematics in Secondary Schools. These lectures consisted largely of a concrete discussion of the problems that arise in actual teaching. This book was planned on similar lines, but the pressure of professional duties, coupled with ill health, made it impossible to carry out this concrete treatment of the subject through the entire extent of secondary school work. The author hopes, however, that all important and fundamental topics have been treated in such detail as to be of real assistance to the inexperienced teacher, to whom the generalities of abstract pedagogy are not only useless, but often meaningless.

The chief object of this book is to contribute towards making mathematical teaching less informational and more disciplinary. Most teachers admit that mathematical instruction derives its importance from the mental training that it affords, and not from the information that it imparts. But in spite of these theoretical views, a great deal of mathematical teaching is still informational. Students still learn demonstrations instead of learning how to demonstrate. This is partly due to

external conditions over which the teacher has no control, but partly also to the fact that a number of teachers have had little opportunity to become acquainted with the details of modern methods of teaching mathematics, and hence largely employ the methods by which they themselves were taught. There can be no doubt that the two excellent American books on the teaching of mathematics — the one by David Eugene Smith, the other by J. W. A. Young — are of great assistance to every teacher, but these books cannot answer many concrete questions on account of the wide range of subjects that they treat of. This book covers a much more restricted field, but does it in greater detail. All references to elementary school work, to history, to the description of movements in other countries, to the material equipment, to mathematical clubs, etc., are excluded, and the discussion of general methods is restricted to their fundamental and most useful phases.

This book is modern in the sense that it attempts to make mathematical instruction less informational and tries to show how to train students in attacking mathematical problems instead of merely making them learn mathematical facts. But it is not modern in the sense that it advocates certain recent fashions which aim to replace the true study of mathematics by applications of doubtful value. While admitting that a certain amount of applied work is very useful and interesting, the author does not believe that the true value of mathematical study lies in its practical utility, and hence cannot admit that the mensuration of parquet

floors or the construction of window designs forms the true end of mathematical study.

In addition to the purely pedagogical discussions, this volume contains certain topics in pure mathematics which, on account of their bearing upon teaching, should be familiar to every teacher. These topics relate principally to the modes of attack (Chapter XV), but also to the foundations of mathematics (Chapter V), to the division of the circle (Chapter XVI), and to a few other subjects. These chapters may be omitted when a rapid survey of the purely pedagogic matters is desired.

Like any other pedagogic book, this volume must necessarily contain a great deal that is obvious and commonplace to the experienced teacher. But to write for the latter only would mean to make this book useless for the prospective teacher to whom the study of mathematical pedagogy is most important.

The author desires to acknowledge his indebtedness to Dr. Joseph Kahn and Mr. W. S. Schlauch for the careful reading of the proofs and for many valuable suggestions.

ARTHUR SCHULTZE.

NEW YORK,
June, 1912.

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THE TEACHING OF MATHEMATICS IN SECONDARY SCHOOLS

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CHAPTER I

CAUSES OF THE INEFFICIENCY OF MATHEMATICAL TEACHING

LOW EFFICIENCY OF SECONDARY SCHOOLS

The present condition of mathematical teaching. — The widespread reform movement for improving the teaching of mathematics, and the increasing interest of teachers in the pedagogy of this subject, seem to be largely due to a general dissatisfaction with the results of mathematical instruction. For, in spite of our pedagogic progress, in spite of the strenuous efforts of our teachers, these results are in general unsatisfactory. Although the apparent results as measured by examinations are often excellent, they are usually not lasting. Students fail to grasp the spirit of the subject, and are often utterly unable to apply their knowledge to advanced work or to practical problems. All who have had an opportunity to test the true mathematical training of the average student a short time after his graduation agree that this training is exceedingly slight.

Remedies proposed. — The conviction that the teaching of mathematics is greatly in need of reform, seems to be almost general. and only when it comes to a dis-

cussion of the causes of the evil, and the remedies that should be applied, does a great diversity of opinion appear.

"Mathematics has outlived its usefulness as a subject of secondary school instruction." — "It is too remote from life to interest the student." — "There is no such thing as mental discipline, hence mathematical teaching has no value," etc. Such are the arguments proposed by men who dislike mathematics, who possibly never had a full understanding of the nature of the subject, and who consequently wish to replace it by some of their pet subjects, such as economics or psychology.

"It is all the fault of the teachers who do not carry out the excellent plans of their superiors, and who do not make students work enough," is an opinion occasionally expressed by school superintendents and principals.

"School mathematics must be made more rigorous," argues a — fortunately decreasing — group of teachers. "If the fundamental notions of limits and incommensurable numbers were taught in a scientific manner, and the slipshod method of assuming things that can be proved were discontinued, then every graduate would understand mathematics, and we would no longer hear that $\sqrt{a^2 + b^2}$ equals $a + b$."

But aside from these laymen or hobby-riding enthusiasts, nearly all teachers of mathematics try to find remedies for the present unsatisfactory conditions, and the cure recommended by most of them is the introduction and study of the applications, along with the pure

science. Apply mathematics, they tell us, teach applications and pure mathematics side by side, make the pure mathematics grow out of its applications, and eliminate, as far as possible, those parts of algebra and geometry that have no immediate practical bearing.

Undoubtedly the general recognition of the fact that the concrete must precede the abstract, and that a great deal of the time-honored mathematical subject matter has but small value, means a great advance in mathematical pedagogy.* It seems doubtful, however, whether this one principle, even if it could be carried out completely, would be sufficient to improve matters thoroughly. It is even doubtful whether under the present conditions *any* change in the subject matter taught could produce a considerable betterment; for the inefficiency of teaching is not confined to mathematics, but appears in nearly all other subjects. The average student within a short time forgets so much of his history, physics, or economics, that it is no exaggeration to say that his permanent knowledge falls far short of the amount studied and the grades attained in examinations. Hence the inefficiency of mathematical teaching cannot be due mainly to the selection of the mathematical subject matter; it must be due to the same general causes that are at work in the teaching of any subject.

In other words, it seems to the author that we have to deal here not with a local, but with a constitutional, disease, and that only an analysis of the general causes that are responsible for the failure of our schools to

* For further discussion of the movement, see Chapter XVIII.

attain their highest efficiency can shed any light upon the problem.

Shortcomings of our schools. — In spite of the undeniably great progress in educational practice during the past decades, a large and increasing number of facts seem to indicate that the results obtained in our schools are not in proportion to the time and labor expended by pupils and teachers. While theoretical pedagogues speak with pride of our modern schools, all unprejudiced persons who have had the opportunity to test the lasting and true knowledge of graduates are rather skeptical about the excellence of these schools. Business men complain that it seems to be harder now than ever to secure boys who can spell correctly, and who know the fundamentals of arithmetic. Teachers in high schools complain that many pupils entering these institutions know very little about grammar school subjects, in spite of their former high examination marks. College teachers are equally dissatisfied with the average results of high school training. In most comments by the press there is an undercurrent of dissatisfaction with the results of our schools.

While many educators try to ridicule such statements and to brand them as gross exaggerations, nobody who has come in personal contact with the students and graduates of our schools can deny that these complaints are well founded. Most students forget, not only the minor and unessential facts, but the most fundamental, the most necessary ones, so completely that the returns for time and labor spent seem to be wholly inadequate.

One may suppose that if the knowledge acquired in our schools is inadequate, possibly the gain in mental power makes up for this shortcoming. But in this respect also the gain is exceedingly small, and principally due to the natural growth of the individual, and not to severe study. Everybody who has had the opportunity to observe young people in their school work cannot have failed to notice the very small use they make of their reasoning power. Not that most students do not possess the necessary intelligence, but they are disinclined to use it, and they do not seem to be aware of the fact that reasoning is often a better means for obtaining an answer than is the thoughtless repetition of words.

The only marked gain is the gain in general culture and refinement. This is due, however, not to the hard study and cramming, but to the general atmosphere of a school; for the lazy student will acquire it equally with the hardest worker. Wherever we search we fail to find a result that entirely justifies the exceedingly hard work and the intense strain to which a high school student is subjected for a term of four years.

For great indeed is the strain to which the young people — those who are conscientious and do all the work required — are subjected, especially in the public educational institutions of our large cities. It is not at all exceptional for high school girls, eighteen or nineteen years old, after more than five hours daily work in school, to spend four or five hours in the preparation

of their home lessons, sitting up late every school night with practically no out-of-door exercise.*

Considering the lasting results only, and not the misleading examination results, it can hardly be maintained that an average secondary school is an efficient machine. It is no exaggeration to say that a fairly intelligent student could get as much culture and true education as a high school graduate by working with an efficient instructor two or three hours daily for a term of four years; although it must be admitted that such a student might not be able to attain spectacular examination results.†

CAUSES OF THE INEFFICIENCY OF OUR SCHOOLS

The overrating of spectacular results.—Unquestionably there are a large number of causes for the inefficiency of our schools. Paramount among these causes, however, seems to be the fact that schools are frequently conducted on the spectacular plan. True results are not sufficiently appreciated; it is "show" and appearance that often are principally aimed at. Undue striving for examination results, crowded courses of study, unnatural and faulty modes of study, promotion of absolutely unfit students, slight appreciation of good

* Every teacher knows that there are a great many pupils who do not work hard because they do not do the required amount of study. There are pupils who, relying upon luck, the help of others, cheating, etc., manage to make their way through school. This unfortunate fact, however, can hardly be used in defense of present conditions.

† This inefficiency of the high schools seems to be fully equaled, if not surpassed, by that of the last years of grammar school, while the work in the more elementary classes appears to be more efficient.

teaching, are all caused by this desire to produce *measurable* results that impress the outside world.

Examinations. — So much has been written on the ill effects of giving too much weight to examinations, that it is hardly necessary to discuss this point in minute detail. Suffice it to say that we all admit that some examinations are necessary, but that ill effects will arise as soon as examinations become the central fact in school life, and especially when they assume a competitive character. As gauges of the work done, as indicators of the efficiency of a particular school, they are unreliable. While exceedingly low examination results may indicate defects, an unusually high average does not at all indicate efficiency of a school, but more often the opposite. Such high examination percentages frequently indicate the employment of wrong methods of teaching and the abuse of both pupil and teacher. The fact that examinations are not proper gauges of the quality of a school has long been recognized in nearly all civilized countries, and hence most of them have attempted to diminish the importance of examinations. Even China has followed their lead and reduced the large number of its examinations. In the United States the teachers as a class are also opposed to the abuse of examinations, but unfortunately those in power frequently think otherwise, and consequently the examination evil has been growing.

Congested courses of study. — It is characteristic of our times that most people consume far more mental food than they can assimilate. It is nothing unusual

for an educated person to read four novels and many magazines a month, to see a play every week, to make frequent visits to concerts, lectures, and art exhibitions, and to devote hours every day to newspaper reading. Everything is done in a hurry, and nothing leaves a lasting impression. Rome is "done" in a day, the Louvre in two hours. There is no time to think, no time to meditate about any of these matters, and general superficiality and mental flabbiness are the result.

One would expect the schools to exert a wholesome influence in opposition to this ever growing shallowness. But far from it; they are the worst offenders. In their anxiety to secure spectacular results, they compel their pupils to do an enormous amount of work in a given time; so much, indeed, that haste and superficiality *must* result. There are schools that finish the whole of plane geometry in six months, and are proud of this feat. There is more taught in many high schools during four years than the average human mind can assimilate in eight.

It is not at all exceptional that a high school student has to master in one evening ten to fifteen pages of history, four pages of geometry, and equally long lessons in two or three other subjects; and this cramming process goes on day after day, year after year. Can any human mind properly assimilate all this material? And do students really accumulate "mental fat," as Spencer claims, or is not the result in 90 per cent of all cases chronic mental indigestion, *i.e.*, utter inability to assimilate any mental food properly? For frequently this

work is done quite mechanically without bringing into play any of the higher powers of the mind. Students simply memorize and usually do not even know that there are other ways than memorizing for acquiring knowledge.

Two ways of studying. — There are two ways of studying new facts. One person who wishes to study a new topic will read over the text again and again, until the words cling to his mind and he can readily repeat any part of the subject involved. Another person will read little, but will meditate upon the subject. He will try to associate the unknown with the known; will attempt to solve, as far as possible unaided, all problems involved, and hence he will look at the subject from all sides. Such a judicious mode of studying is a far slower process than memorizing, but it leads to lasting and full understanding and true knowledge of the subject. The first method, memorizing, is a perfectly proper method for the most elementary things, but leads to absolutely no results in the more advanced subjects. The multiplication table, words of a foreign language, spelling of words, etc., can be fully mastered by mechanical memorizing, but it is absurd to study geometry, physics, or philosophical subjects in the same manner.

Most students are familiar only with the first method. They never go through the transition from childish memorizing to judicious study. However, by their ability to repeat words, they often deceive others and themselves into the belief that they have mastered their

studies. The deceptive character of knowledge thus acquired is demonstrated by the rapidity with which it is lost. In a very short time everything studied, essential and unessential, utterly fades away. On the other hand, the person who studies judiciously receives such a strong impression of the essentials, that he rarely forgets, and even if he does, he can frequently reconstruct the missing data. His previous thinking has given him a strong framework of facts, upon which all minor data are readily assembled.

Schools encourage memorizing and neglect reasoning. — In the preceding paragraph it was pointed out that mechanical memorizing is a perfectly proper method of studying the most elementary, the most fundamental facts, which are of frequent application.* This is possibly the reason why the teaching is far more effective in the lower grades than later on. In more advanced work the very nature of the subjects makes mere memorizing ineffective.

Our high schools, however, not only encourage memorizing, but sometimes almost force the student to adopt this as the only mode of study, for only by memorizing can he hope to satisfy the immediate demands of the school.

The daily rations of mental food that the student has to swallow give him no choice; there is no time for

* Such subjects, however, must be made so familiar to students by frequent repetition that forgetting becomes almost impossible. In many schools, topics of this kind, however, are treated just as hastily as everything else; hence memorizing even in its legitimate place is rendered ineffective.

thought, for meditation, for judicious study; he *must* memorize. Moreover, the character of the studies leads him to mechanical work, for in spite of the vigorous denials of our pedagogues, the greater part of the curriculum is informational. It is knowledge and not power that is emphasized in most of the studies, and even subjects which by their very nature should be mastered by thinking are often made informational. For the informational method produces much quicker and more spectacular results than the slow judicious mode of study. What a fine display of learning students can make if they have been cramming conscientiously! How high the percentage they can secure in examinations! True, the after effects are sad, but who cares? As long as the boy can talk glibly about complex economic problems in terms which he does not understand, we are satisfied. What does it matter, that a year later he has not the remotest inkling of the subject, that he cannot discuss intelligently the simplest new problem that may arise!

Can we wonder that under such conditions the student never breaks away from his mechanical way of studying that he acquired in the elementary school? And can we wonder, too, that the results of our teaching become inferior in the higher grades of the grammar school, and especially so in the high school?

This excessive use of memorizing, and the neglect of the cultivation of the reasoning power, are possibly the worst effects of the spectacular idea upon which our schools are largely built.

Star students. — It must be admitted that there is a small group of students who possess memories of extraordinary retentive power, who can memorize and retain anything from geometry to metaphysics. Notwithstanding the "mental fat" that such students accumulate, they ought to be pitied, for they rarely cultivate their higher mental powers. To them memory is such a convenient tool, that it is used almost exclusively, while the other faculties of the mind atrophy. Such people rarely become thinkers, they are without originality, they do not produce, but only re-produce, ideas. In school, however, pupils of this class excel; they are the star students, the pride of their teachers and parents.

Perhaps this is the reason why star students so seldom fulfill the expectations of their friends. They hardly ever excel in the ordinary pursuits of life, or become able scholars. Frequently, however, their apparent success in studying leads them into the teaching profession, and they then expect every individual to possess the same freakish memory which they have. Such men, especially if they reach the higher positions, usually do all in their power to intensify the cramming conditions of our schools.

On the other hand, students of great ability have sometimes poor memories for words and are therefore regarded as hopeless dunces by their teachers. There is no lack of examples of great men who were considered very dull boys by their teachers, simply because they did not possess that parrot quality which is so highly appreciated in school.

The absurdity of making the memory the chief standard for measuring a student's ability cannot be overemphasized. Which is more useful to a person in life, the knowledge of a great many facts, or mental power? the ability to repeat other people's thoughts, or the ability to think for himself?

EFFECTS OF THE GENERAL CONDITIONS UPON MATHEMATICAL TEACHING

In view of the conditions in many primary and secondary schools, it is not surprising that mathematical teaching produces poor results. The entire atmosphere of some schools is so opposed to the true mathematical spirit that true teaching and true studying of the subject are almost impossible. In many schools good mathematical teaching is not understood, not appreciated, not wanted by the authorities; while the fictitious but showy results of the drillmaster are highly commended. Take, in addition, the lack of time, the preparation for difficult examinations, the poor preparation of the students, their firmly rooted habits of memorizing, and their mental inertia due to years of mechanical work, and we cannot be surprised that teachers — against their better knowledge — use faulty methods of teaching. Instead of real mathematical work, a cramming process is employed that is not only useless, but positively harmful to the students; for mathematics taught as an informational subject is exceedingly tiresome and injurious to the mind.

No other subject suffers so much and becomes so valueless as mathematics, when treated by mechanical

modes of study; and, on the other hand, no other secondary school subject is so admirably adapted to a judicious mode of study as mathematics. In fact, this characteristic constitutes one of the chief values of this study, and must be constantly considered when we wish to determine the aims and methods of mathematical teaching.

CHAPTER II

THE VALUE AND THE AIMS OF MATHEMATICAL TEACHING

THE PRACTICAL VALUE OF MATHEMATICS

Classification of the advantages of mathematical teaching. — In order to determine with precision the methods of teaching any subject, it is necessary to arrive at a clear understanding of the reasons for teaching it. This inquiry is particularly important in mathematics, since the aims and pedagogical advantages of this study differ widely from those of most other subjects.

The many reasons that may be given for teaching mathematics are usually classified under two heads, viz.: those based upon (1) the practical value of mathematics, or (2) the culture it imparts.

The practical value of mathematics is very great. It is indeed no exaggeration to assert that our whole modern civilization owes its peculiar stamp indirectly to mathematics. Modern thought and modern life owe their character to the great progress of the exact sciences, and to the wonderful development of the technical arts. These two in turn are closely connected with, and based upon, mathematics.

Importance of mathematics in science. — A science becomes exact, when it advances from the formation of mere qualitative relations to quantitative laws, and

thereby becomes accessible to mathematical investigations. Hence Kant said, "A science is exact only in so far as it employs mathematics." The knowledge of the deflection of a ray of light entering from one medium into another of different density was of small value until the quantitative law of refraction was discovered. Thereby all problems of dioptrics became mathematical problems, and the entire art of making optical instruments was put upon an exact scientific basis.

Similarly the discovery of the laws of motion and the law of gravitation transformed all problems of celestial mechanics into problems of mathematics, and, owing to the exactness of mathematics, this branch of astronomy in a short time reached an amazing degree of perfection.

Astronomy and physics are the most exact sciences, and hence are the best illustrations of the usefulness of mathematics. But chemistry and geology, economics and physiology, all use mathematics. Even psychology, if it accepts the Weber-Fechner law, cannot dispense with the help of mathematics.

Mathematical knowledge is indispensable for the understanding of the phenomena of nature, and no one without mathematical scholarship can hope to advance far as an investigator in most of the exact sciences.*

* The most useful of all mathematical devices are differential equations, since almost any branch of physical science leads to these. The state of a physical system is the absolute consequence of the state immediately preceding it, a relation that necessarily leads to a differential equation. The symbolism of differential equations is better adapted to represent the fundamental phenomena of nature than is any other

Influence of mathematics upon life. — This is the age of machines. The production and distribution of every necessity or luxury partly depend upon the technical sciences which owe their perfection to their exact mathematical basis. "Our entire present civilization," says Professor Voss, "as far as it depends upon the intellectual penetration and utilization of nature, has its real foundation in the mathematical sciences." Engineering, architecture, navigation, railroad building, and surveying are more or less based upon mathematical foundations.

Moreover, this influence of the exact sciences, and hence of mathematics, is increasing so rapidly that a nation which would base its industries upon purely empirical rules, to the exclusion of scientific methods, would be hopelessly handicapped, and left behind in the struggle for commercial and industrial supremacy.

Value of mathematical knowledge to the individual. — It would be an error to infer, from the great usefulness of mathematics to our civilization, an equal practi-

means. These symbols not only enable us to state laws which otherwise could not be stated at all, but they often make it possible to express relations between physical quantities whose true inward nature is entirely unknown. Fresnel's theory of light, which attributes light to movements of the ether, leads to differential equations which give a satisfactory explanation of most optical phenomena. To-day most physicists accept Maxwell's electro-magnetic theory of light. But while this theory has changed our notions of the physical nature of light, it has left Fresnel's differential equation unaltered. Mechanical explanations of the phenomena change, but the differential equations remain unaltered. Differential equations are such powerful instruments for exact physical investigations, that we can understand Riemann's saying: "Exact science exists since the discovery of differential equations."

cal usefulness to every individual. The percentage of students who are likely to have practical use for mathematics, after leaving school or college, is certainly small. The majority of business or professional callings require no algebra, geometry, or trigonometry, and even the professions which use these subjects do so to a much smaller extent than is generally supposed. There are navigators, surveyors, and engineers who make their calculations in an almost mechanical manner, without having perfectly clear notions of the underlying mathematical principles. Only for those few men who become original designers and investigators is true mathematical skill and knowledge indispensable. Still, mathematics has some practical value for all students, and even to an extent greater than many other high school subjects.

If mathematics, however, had no value as a mental discipline, its teaching in the secondary schools could hardly be justified solely on grounds of its bread-and-butter value.

THE DISCIPLINARY VALUE OF MATHEMATICS

General remarks. — The principal value of mathematical study arises from the fact that it exercises the *reasoning power more*, and claims *from the memory less*, than any other secondary school subject. The study of mathematics should result in the development of *power*, rather than in the acquisition of facts. Not he who knows a great many mathematical facts is a good mathematician, but he who can apply these facts intelligently,

who can discover facts that are new to him, and who can reconstruct those which he has forgotten.

It is power and not knowledge that furnishes the true test of mathematical ability, and if the power is acquired, then — and only then — will the knowledge follow as a natural consequence. Mathematical instruction in a secondary school is — or rather should be — principally a systematic training in reasoning, and not an imparting of information.

Of course similar claims are made for nearly all other subjects, but a closer inquiry will show that for mathematics they are really justified. The reasoning in mathematical work is of a peculiar kind, possessing characteristics that make it especially fitted for training the minds of the students. Some of these characteristics are the following:

1. *Simplicity.*
2. *Accuracy.*
3. *Certainty of results.*
4. *Originality.*
5. *Similarity to the reasoning of life.*
6. *Amount of reasoning.*

Simplicity. — It is a well-known principle of physical training that too severe exercises are not only useless, but often harmful to the beginners. Similarly, simple mental exercises are much better adapted to the training of the mind of the young, than very hard ones. Mathematics allows an almost perfect grading, commencing with exceedingly simple work, and leading the student

by degrees to harder and harder problems. It is difficult, for instance, to imagine anything easier than the simplest exercises in geometry. To prove, let us say, the equality of two triangles, the student has to examine six pairs of homologous parts, and to try to find reasons for the equality of three of them. And how simple these reasons are: The hypothesis, an axiom, or one of of the 4 or 5 preceding theorems. How perfectly definite the given facts, the method, and the required result! How few given data the student has to keep in mind, and how few facts he has to know in order to discover the reasons! Contrast with this the reasoning necessary for writing an argument in English, — multiplicity of known facts, indefinite character of the given data and methods to be used, uncertainty of results.

Accuracy. — Every teacher knows how many students lack precision of thought and expression, how many are unable, or do not try, to understand the precise meaning of a question, how many speak before even attempting to think. While it seems that students can get along fairly well in other subjects with such methods, they cannot do so in mathematics. In this subject, the mere repetition of words or phrases will not hide the ignorance of the pupil. The student must think accurately, he has to speak accurately, to master mathematics.

Certainty of results. — Any piece of mathematical work is either right or wrong, and it is usually a very simple matter to find out whether or not it is right. Certainly there can be no difference of opinion between

student and teacher as to the final result. A student who has discovered a geometric original, or has solved an algebraic problem, and verified his answer, knows that he is right, and therefore is conscious of having accomplished something. This sensation of having definitely overcome a difficulty is to the normal pupil a source of pleasure, a pleasure which increases with the conquered difficulty. Compare with this the reasoning a student has to do in a philosophical, political, or economic subject. After the expenditure of much time and labor, there follows uncertainty of result, possible difference of opinion between teacher and pupil, perhaps even a disagreement of the accepted authorities.

Originality. — Mathematical reasoning done by students is entirely original thinking, and not the reproduction of ideas previously heard or read. This cannot be said of other school subjects that claim to appeal mainly to the student's reasoning power. Thus, a student may apparently reason ably in working out an economic problem, while actually the bulk of his answer is taken — consciously or unconsciously — from his memory. Topics of such general character are discussed so frequently in daily papers, magazines, books, and in the family circle, that he who has most opportunities in this direction, or he who has the best memory, is often considered a good thinker, although he may be a dull-witted person.

Similarity to the reasoning of daily life. — While nobody questions the value of mathematical training for scientific work and rigid logical deductions, it is often

asserted that mathematical thinking is of an entirely different order from the kind used in human affairs, and that consequently mathematical training has no practical value.

It is undoubtedly true that the mental qualities cultivated by mathematical study alone are not sufficient to insure ability for solving practical problems; but on the other hand, it is evident that without these qualities one can hardly hope for success in the affairs of life. Clearness and exactness of thinking are just as necessary in daily life as in mathematical study.

The person who undertakes an industrial or commercial venture must possess a clear idea of the existing conditions and of his aims, — in other words, he must have a firm grasp on the situation; just as a student in mathematics has to recognize the hypothesis and the conclusion. Then — precisely as the student of geometry — the business man has to consider the various means at his disposal; he has to examine each, to eliminate those that are unfit, and to weigh and to compare the others. In all steps he must have a clear notion of the situation, of the means to be adopted, of the end to be reached. Confusing the data and random guessing will produce in business no better results than in mathematics.* More than one business man has testified that he owes his success in life to the habits of exact thinking which he formed when studying mathematics.

* For an elaborate presentation of this point see Young's Teaching of Mathematics.

Bulk of mathematical work is reasoning. — Mathematics appeals more to the reasoning power, and less to the memory than any other high school subject. This is particularly true of geometry. Here — if the subject is properly taught — nearly everything is reasoning. The few facts to be known are so palpable as to require no special memorizing. The equality of vertical angles, the equality of equidistant chords, etc., are facts that can be remembered without cramming. Even more complex propositions, by constant application, soon become familiar to the student.

The rest is — or should be — reasoning. After proper training in exercise work, the regular propositions soon become natural consequences of general methods — not to be memorized, but to be discovered, and to be reconstructed when forgotten.

Algebra does not make quite so good a showing as geometry. But if its purely formal, manipulative features are not extended farther than necessary for future work, very few facts have to be remembered in algebra, — incomparably fewer than in Latin, history, or French. Moreover, these facts are connected logically and can be reconstructed if forgotten, — a thing generally impossible in a language or other informational subject.

While it must be admitted that some of the thinking of elementary algebra is of the same inferior order as that used in the study of Latin or Greek, namely, the mechanical application of a known rule, this is not true of all the work. There is a large field for good original

thinking in the work on reading problems and in formal work of more advanced character.

Every normal youth can study mathematics. — Under fair conditions every normal pupil can easily comprehend the simple reasoning of mathematics, provided the subject is presented properly, and provided his mind has not been thoroughly dulled by an excess of mechanical study. The small minority that really cannot understand mathematics under any condition does not consist of able pupils, and it is not likely that such students — as is sometimes asserted — excel in other studies. How can a pupil excel in English or economics who frequently uses his conclusion for his proof, demonstrates his hypothesis, jumps at conclusions without plan or reason, and cannot concentrate his mind upon anything, however simple?

Elementary mathematics requires nothing but the plainest common sense; and the story of the special brains needed for mathematics, as far as elementary work is concerned, is a myth.

Denial of all mental discipline. — Some psychologists claim that there is no such thing as general mental discipline, that the disciplinary value pertains only to the subject studied, or to one of similar content, and that consequently mathematical study increases the reasoning power for mathematics only.

It cannot be denied that there is a little truth in the first part of this assertion, and that this theory has produced some reaction against the practice of defending any pedagogical absurdity on grounds of "mental dis-

cipline." But on the other hand there is a tendency among the sensational pedagogues to exaggerate and to generalize too sweepingly. Pedagogy and psychology are not exact sciences. Their results are only approximately true, and cannot be applied in the same rigorous fashion as those of mathematics or physics. If we attempt to apply them to complex problems, the limits of error are likely to become so large as to invalidate the entire results. Conclusions reached by such methods need constant verification, and must be modified if found to be contradictory to experience.

Precisely this thing happens in this widely advertised discipline theory, when we apply it to mathematical teaching. Every mathematical teacher of experience has seen cases which disprove this theory. It is a common experience to see a pupil in the upper grades suddenly wake up to the meaning of mathematics, and thereby change his attitude towards study in general. Pupils who were indifferent and apparently without ability become active and intelligent students, interested and capable, not only in mathematics but in other studies. (These studies, however, it must be admitted, are sciences, not languages.) The question is similar to that of the value of physical training — although it is no longer the fashion to use this parallel — for the pursuit of some physical labor. Baseball playing may not be a direct preparation for a particular physical labor such as hod-carrying. Still a man who has strengthened his body by ball playing would be in a better position to take up such a task, than he who has never exercised at all.

Then we have the negative evidence. If this doctrine were true, the harm done by unpedagogic training would not extend beyond the subject or subjects that caused it. But every day we see that the results caused by bad training are general. The dullness acquired in some of our schools is absolutely general; it relates to any subject, old or new. In other words, one or several subjects may disqualify the student for the study of all subjects.

If we should accept the theory that the general mental caliber of the students is not improved by study, it would undoubtedly be best to close all schools after the fourth or fifth year of the grammar school, since the *knowledge* gained afterwards is not worth the trouble.*

MINOR FUNCTIONS OF MATHEMATICAL STUDY

Some of the minor advantages of mathematical study may be briefly stated as follows:

1. **Development of the power of concentration.** — Very few young people seem to be able to concentrate their minds for even a few minutes upon one idea. This is a faculty, however, which can be acquired, and mathematical study is admirably adapted to develop it.

* This theory of mental discipline is closely related to the physiological theory of the localization of mental functions. The latter theory assumes that the mind consists of a number of independent "faculties," each of which has a definite localization in a region of the brain. These views form the basis of phrenology, but they have been generally abandoned by physiologists. (See Loeb, Physiology of the Brain, pp. 259–263.)

2. **Development of the constructive imagination or the inventive faculty.** — Far from being a dry science requiring pedantic accuracy and little imagination, true mathematical work consists in inventing, in finding something that is unknown to the worker; and in this, success is impossible without the use of the creative powers of the mind. Solving a geometric problem and making an invention are very similar processes, the chief point of difference being usually the greater simplicity of the geometric problem. To the student, the solving of a difficult problem is a discovery; and consistent training in such work develops those faculties that lead to discovery and invention.

3. **Growth of mental self-reliance.** — Young students, as a rule, rely too much upon facts taken from books or some other authority, and too little upon their own faculties, a trait which shows that they have no confidence in their own mental powers. Their former training has led them to believe utterly in authority, and especially to think that all knowledge depends upon authority. Some people retain through life the habit of placing authority above common sense and reason. Especially in educational circles is this affliction very common.

4. **Development of character.** — Mathematical study trains the students in *systematic and orderly habits*, and the pleasure connected with the successful conquering of a difficulty stimulates the *will power*. It has also been claimed that dealing with a subject that is absolutely true, that rejects and shows up any error, is bound to increase respect for *truthfulness and honesty*.

5. **Increased ability to use English correctly.** — The difficulties which many students have in attempting to express their own ideas in their mother tongue are quite apparent in geometry, and on the other hand few subjects are so well suited as geometry for curing this evil. Geometric work possesses three qualities which are necessary for such work, viz. originality of the idea, simplicity of the terminology, and comparative ease with which precision of expression can be reached. The ability to understand English, especially of the rather difficult kind used in scientific or philosophical discussion, is also greatly improved by the study of mathematical texts.

6. **Increase in general culture.** — An acquaintance with the fundamental facts and methods of mathematics seems to be necessary for general culture. A science that is closely interwoven with most mental achievements of the race, that is found in all civilizations, that represents the most finished types of exact thinking, cannot be ignored by the man of culture.

A person unfamiliar with the elements of mathematics cannot fully comprehend the simplest facts of astronomy, he is not able to read and to grasp the accounts of the wonderful discoveries and inventions of our time.

Summary. — If the student becomes properly initiated into the spirit of mathematical work, and pursues his studies with interest, some of the principal results may be briefly stated as follows: He will be led to an intelligent use of his reasoning faculties, and will recognize

that thinking is more effective in study than is memorizing. He will be led to a judicious mode of study in general, and acquire mental self-reliance and independence. He will take more interest in, and thereby derive more pleasure from, his studies than ever before.

It is needless to say that in most cases these results are not obtained. As pointed out before, however, this is not due to the nature of the subject, but to the perversion of its true spirit by those who control our schools, and who frequently have no understanding of the peculiarities of mathematics.

The fundamental principle of mathematical teaching. — The acceptance of the foregoing views on the purpose of mathematical teaching must more or less influence all principles of mathematical pedagogy. Among the numerous consequences, one is of such great importance, that it is no exaggeration to call it the fundamental principle of mathematical teaching, viz.:

Mathematics is primarily taught on account of the mental training it affords, and only secondarily on account of the knowledge of facts it imparts. The true end of mathematical teaching is power and not knowledge.

CHAPTER III

METHODS OF TEACHING MATHEMATICS

CLASSIFICATION OF METHODS

No attempt will be made to give in this book an exhaustive discussion of all methods that have any bearing upon mathematical teaching.* The essential methods, however, — those which have a substantial bearing upon the work of the teacher, — will be discussed rather fully. They are the following:

1. The Synthetic and the Analytic Methods.
2. The Inductive and the Deductive Methods.
3. The Dogmatic and the Psychological Methods.
4. The Lecture and the Heuristic Methods.
5. The Laboratory Method.

Each of these methods refers to a different phase of presentation of the subject, and consequently they do not exclude one another.

THE SYNTHETIC AND THE ANALYTIC METHODS

Description of the two methods. — Synthetic methods lead from the known to the unknown, while analytic methods proceed from the unknown to the known. In geometry a synthetic proof starts from the hypothesis and ends with the conclusion, while an analysis leads from the conclusion to the hypothesis. In a synthesis

* For greater detail the reader is referred to Young's Teaching of Mathematics.

we say: A is true, therefore B is true, and therefore C is true. In an analysis we reason: C is true if B is true, and B is true if A is true. But A is true, hence C is true.

The demonstrations given in the textbooks of geometry are nearly all synthetic, while analytic proofs, in most textbooks, are entirely omitted.

Examples. — The true character of the terms *analytic* and *synthetic*, as used in elementary mathematics, will possibly be best explained by a number of concrete examples.*

Ex. 1. If $$a : b = c : d,$$
then $$ac + 2b^2 : bc = c^2 + 2bd : dc.$$

Synthetic Proof.

$$\frac{a}{b} = \frac{c}{d}.$$

Adding $\frac{2b}{c}$ to each member,

$$\frac{a}{b} + \frac{2b}{c} = \frac{c}{d} + \frac{2b}{c}.$$

Simplifying, $$\frac{ac + 2b^2}{bc} = \frac{c^2 + 2bd}{dc}. \qquad \text{Q.E.D}$$

Analytic Proof. The identity

$$\frac{ac + 2b^2}{bc} = \frac{c^2 + 2bd}{dc}$$

would be true if $$(ac + 2b^2)dc = (c^2 + 2bd)bc.$$

This would be true if

$$ac^2d + 2b^2cd = bc^3 + 2b^2cd,$$

or if $$ac^2d = bc^3,$$

or if $$ad = bc.$$

But $$ad = bc. \qquad \text{(Hyp.)}$$

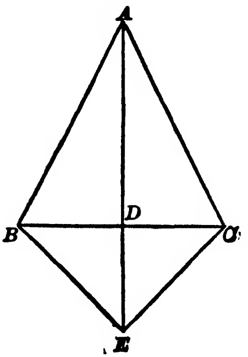
Therefore, $$ac + 2b^2 : bc = c^2 + 2bd : dc.$$

* For additional examples see Chapter XI.

The *synthetic proof* is shorter, more elegant, but one does not see why $\frac{2b}{c}$ was added to each member, although this operation is justified by the result. The *analytic proof* is lengthy and almost awkward, but there is no doubt why this sequence of steps was taken. The synthetic proof is a special device, the analytic is based upon a general method. If the student should forget, it would be much easier for him to reconstruct the analysis than the synthesis.

Ex. 2. The line that joins the vertices of two isosceles triangles having a common base is perpendicular to the common base.

The *synthesis* of the proposition is the proof commonly given in the textbooks. We prove first the equality of the triangles ABE and ACE, then the equality of triangles ABD and ACD. From the resulting equality of angles BDA and CDA follows the conclusion.



Analysis. (1) Prove $\angle BDA = 90°$.

To prove that an angle is a right angle, we usually prove that it is equal to its supplementary adjacent ones (Chapter VIII, Method IV), therefore we have to prove:

(2) $\angle BDA = \angle CDA$.

The equality of two angles is usually proved by means of two equal triangles (Chapter VIII, Method I), therefore we must prove:

(3) $\triangle BDA = \triangle CDA$.

Since we cannot find enough equal parts to prove the equality of the two triangles directly, we select first another pair of triangles

whose homologous parts will supply the missing equalities (Chapter VIII, Method III). Therefore prove:

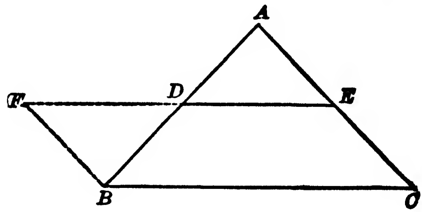
(4) $\triangle ABE = \triangle ACE$, an equality which is easily established.

The reduction of the conclusion to a simpler or more easily proved statement is accomplished by an inquiry into the various means of proving the conclusion. Hence we do not simply say, "The lines are perpendicular if the two triangles are equal," but ask for the means for proving the perpendicularity of two lines.

Ex. 3. The line joining the mid-points of two sides of a triangle is parallel to the third side and equal to one half of it.

The *synthesis* is the proof commonly given in textbooks.

Analysis. (1) The usual method for proving that one line (ED) is one half of another (BC) is to double the smaller.* Hence we produce ED by its own length to F.



(2) Prove $EF = BC$, and $EF \parallel BC$.

There are several methods for proving the equality of lines, and several others for proving parallelism of lines. Only one, however, proves equality and parallelism simultaneously, viz. the one based upon a parallelogram. Hence we attempt to

(3) Prove that $BCEF$ is a ▱.

There are again various methods of demonstrating that a figure is a parallelogram, but since nothing whatever is known in regard to BC and EF we have to use the sides BF and CE exclusively, *i.e.* we have

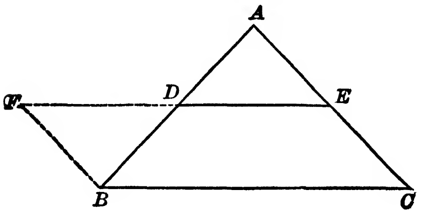
(4) To prove (*a*) $BF \parallel CE$ and (*b*) $BF = CE$.

* Chapter XI, Method VII.

(*a*) The fundamental method for demonstrating parallelism of lines rests upon the equality of a pair of alternate interior angles. Hence we have to prove

(5) $\angle A = \angle DBF$.

The usual method for demonstrating the equality of two angles makes use of the equality of triangles, or we have to prove



(6) $\triangle ADE = \triangle DFB$.

The equality of these triangles is easily established, and hence the parallelism of BF and CE follows. To prove the equality of these two lines we have to remember that $CE = AE$. The equality of AE and BF follows easily from the triangles considered in (6).

This may appear difficult and artificial to a person not familiar with analyzing in general, and not accustomed to consider the various "means" for demonstrating certain geometric facts. But it is on the other hand obvious that this analysis gives a reason for taking each step, and that a student acquainted with the various "methods" will in most cases find this solution. The synthesis, however, consists of a number of steps, whose correctness we see, but for whose sequence we have no reason whatsoever. In the synthesis we do not see why ED is produced by its own length, why we demonstrate the equality of triangles FBD and EAD, etc.

Ex. 4. The sum of any two face angles of a triedral angle is greater than a third face angle.

Analysis. (1) To prove $\angle AVB + \angle BVC > \angle CVA$.

When we compare the sum of two angles (AVB and BVC) with a third angle CVA, we either construct

(*a*) The sum of AVB and BVC, or

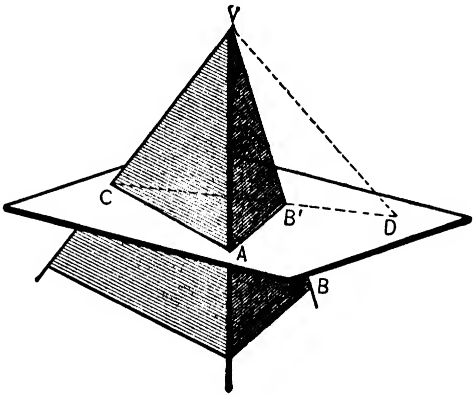
(*b*) The difference of CVA and AVB. Method (*b*) leads to the demonstration that is generally given in textbooks, and is quite analogous to (*a*). Hence only (*a*) will be analyzed.

In plane BVC draw VD so that $\angle DVB = \angle AVB$, then we have to prove that

(2) $\angle DVC < \angle AVC$.

There are three fundamental methods for proving the inequality of angles,* but since the angles lie in different triangles, we must use the proposition of two triangles that have two sides equal, but the third sides unequal.

Obviously VC is common, and to obtain the other pair of equal sides we make $VD = VA$ and pass a plane through A, D, and C intersecting VB in B', then CVA and DVC are two triangles that



have two sides of one equal to two of the other, and it remains to prove that

(3) $CA < DC$.

But $CA < CB' + B'A$, and since $CD = CB' + B'D$, the theorem would be proved if we could show that

(4) $B'A = B'D$.

But this is easily established by means of equal triangles.

* See Chapter X, Unequal Lines and Angles.

Value of the two methods. — A synthesis shows that every step is true, but does not explain why this step was taken. A synthetic proof *convinces* the reader that the fact to be demonstrated is true, but does not reveal to him the real plan of the demonstration, does not tell him why this sequence of arguments was selected. Proofs are not discovered by the synthetic methods, and if forgotten, synthetic demonstrations are most difficult to reconstruct. But synthetic proofs are usually short and elegant, and are in place when no pedagogical conditions need to be considered.

An analysis, on the other hand, is lengthy and not elegant, but it is the only method that accounts fully for each step of demonstration. It is the only method by which students can hope to discover proofs, or to rediscover them after they are forgotten. *Analysis is the method of discovery, synthesis the method of concise and elegant presentation.*

Hence, students in secondary schools should be made to discover demonstrations by analysis, but after this has been accomplished, the proof may be represented synthetically. Exclusive synthetic teaching may be excusable when used with university students, who are able to analyze for themselves, but even there bad results will frequently follow.*

* Some authors, *e.g.* Gauss, are exceedingly difficult to read, because their demonstrations are frequently very concise syntheses, with no indications of the analytic steps by which the author arrived at his conclusions.

THE INDUCTIVE AND THE DEDUCTIVE METHODS

Description of the two methods. — The *inductive* method leads from the particular to the general, from the concrete to the abstract, while the *deductive* method derives particular truths from general truths, concrete facts from abstract facts.

Any syllogism is a good example of deductive reasoning, *e.g.*:

Vertical angles are equal.
$\angle A$ and $\angle B$ are vertical angles.
Hence, $\angle A = \angle B$.

On the other hand, any conclusion drawn from experience, any general law derived from a number of experiments, is obtained by induction, *e.g.* "Whenever a body of gas is compressed, its temperature rises, hence all gases will become hotter if compressed."

Inductive reasoning is not absolutely conclusive, it only establishes a certain degree of probability, which increases with the number of facts observed. Hence it cannot be used for exact mathematical demonstrations,* but it can be employed for *finding* mathematical facts.

Mathematical facts discovered by induction. — Such discovery is best illustrated by concrete examples, *e.g.* to find the sum of the first n natural numbers, let us find this sum from $n = 1$ to $n = 5$, and let us compare (*i.e.* divide) these sums with n. We obtain then the following table:

* On the inductive elements in mathematics see Chapter IV.

n	Series	Sum	Sum compared with n
1	1	1	1 or $\frac{2}{2}$
2	1 + 2	3	$\frac{3}{2}$ or $\frac{3}{2}$
3	1 + 2 + 3	6	2 or $\frac{4}{2}$
4	1 + 2 + 3 + 4	10	$\frac{5}{2}$ or $\frac{5}{2}$
5	1 + 2 + 3 + 4 + 5	15	3 or $\frac{6}{2}$

Since the quotient is $\frac{2}{2}$ for 1, $\frac{3}{2}$ for 2, $\frac{4}{2}$ for 3, $\frac{5}{2}$ for 4, etc., it is very likely $\frac{n+1}{2}$ for n terms, or the sum of the first n natural numbers is probably $\frac{n(n+1)}{2}$.*

A *deductive* method of obtaining this formula is the substitution of $f(x) = x^2$ in the general formula

$$\sum_1^n [f(x) - f(x-1)] = f(n) - f(0).\dagger$$

Similarly the next table shows us how to find the sum of the squares of the natural numbers by induction. Here, however, the comparison with n does not easily lead to an answer, but the comparison with the sum of the first n natural numbers gives $\frac{3}{3}$ for 1, $\frac{5}{3}$ for 2, $\frac{7}{3}$ for 3, $\frac{9}{3}$ for 4, $\frac{11}{3}$ for 5.

Hence we reason that it will be $\frac{2n+1}{3}$ for n, or the required sum is probably $\frac{n(n+1)(2n+1)}{6}$.

* To demonstrate fully facts obtained in this manner, additional methods are necessary. In the above example, such a method would be the so-called "mathematical induction," which, however, does not signify induction in the ordinary philosophical sense.

† See the author's Advanced Algebra, p. 542.

n	SERIES	SUM	SUM COMPARED WITH $\frac{n(n+1)}{2}$
1	1	1	1 or $\frac{3}{3}$
2	1 + 4	5	$\frac{5}{3}$ or $\frac{5}{3}$
3	1 + 4 + 9	14	$\frac{7}{3}$ or $\frac{7}{3}$
4	1 + 4 + 9 + 16	30	3 or $\frac{9}{3}$
5	1 + 4 + 9 + 16 + 25	55	$\frac{11}{3}$ or $\frac{11}{3}$

Deductive and inductive sequence. Pedagogically the term "inductive" relates principally to that sequence which places the concrete before the abstract, the special example before the general formula, while the deductive uses the opposite succession.

Thus it would be deductive sequence first to derive the formula for the square of a side of a triangle opposite an acute angle ($a^2 = b^2 + c^2 - 2\,cp$) and to solve *every* numerical example by substitution in this formula. It would be inductive sequence first to give a number of numerical examples before the general theorem is attacked.

To simplify expressions of the type $\sqrt{a+\sqrt{b}}$ by the deductive method, we would first derive the general formula

$$\sqrt{a+\sqrt{b}} = \sqrt{\frac{a+\sqrt{a^2-b}}{2}} + \sqrt{\frac{a-\sqrt{a^2-b}}{2}}$$

and solve every numerical example of this type by substitution. Using induction, we would *first* solve a number of numerical examples as $\sqrt{5+2\sqrt{6}}$, $\sqrt{9+\sqrt{56}}$ etc., and only afterwards, possibly, attack the problem $\sqrt{a+\sqrt{b}}$.

Value of the two methods. — The solution of mathematical problems, by previously demonstrated formulæ, is in general a very concise and practical method. It is much shorter to memorize a formula for the median of a triangle than to find the numerical value in every concrete example by starting from the general median proposition. Hence teachers as well as textbooks frequently give deductive methods a prominent place, sometimes almost to the exclusion of all inductive work.

On the other hand, it is very difficult for a beginner to understand an abstract piece of mathematical work, if not preceded by a number of concrete instances. Can students really understand the derivation and the meaning of the formula for the number of permutations of n things, taking r at a time, if the problem is not preceded by a number of concrete cases? Can they even know what a permutation is without concrete illustrations? Abstract ideas are the result of concrete experiences, and only after a number of concrete cases are understood can abstract generalizations be successfully attacked.

Moreover, purely deductive methods require a formula for every type of mathematical problems, and the extensive use of such methods demands the memorizing of a great many formulæ. After forgetting these formulæ — and the forgetting takes places very rapidly — the student is utterly helpless. Who remembers Cardan's formula? Pupils trained to attack problems inductively need not rely upon this formula; they can apply Cardan's method without knowing the formula "by heart." *

* See the author's Advanced Algebra, p. 495.

The preceding arguments, however, do not involve the abandoning of all deductive methods. On the contrary, for certain types of work, deduction is most important, although it should always be preceded by induction. Thus, for all important problems of fundamental character, formulæ should be memorized. For instance, the formulæ for the roots of a quadratic equation, the formulæ for progressions, the binomial theorem, etc., should be thoroughly memorized and applied. Hence we cannot use one of these methods to the exclusion of the other, and we may summarize as follows:

1. Apply inductive methods whenever there is an opportunity.

2. Use deductive methods also, but only after attacking the subject inductively.

3. Deduction, and the consequent memorizing, should be restricted to the most important cases.*

THE DOGMATIC AND THE PSYCHOLOGICAL† METHODS

The dogmatic method. — The dogmatic method makes rigor the chief desideratum of mathematical study, while

* Lately there has been a tendency to teach general theorems at a very early stage of the work. A typical instance of this kind is the factor theorem. The reasons advanced above make it undesirable to study general theorems at too early a stage of the work. For, as a rule, the students do not understand, and hardly ever appreciate, such theorems. Let the student first become familiar with the use of simple tools, before we let him use highly refined instruments. No general theorem should be taught before the student recognizes the necessity for such a proposition, and before he is aware of the difficulty of the special cases.

† Sometimes called the genetic method, a term which is, however, frequently used in a different sense.

the psychological method advocates rigor only so far as the average capacity of young students justifies it. The dogmatists claim that the value of mathematical teaching rests mainly upon its extreme exactness, that any deviation from the absolute standard of rigor will defeat the very purpose of the entire work, and will necessarily lead to slipshod thinking. Indeed, there are teachers who lay the inefficiency of mathematical teaching mainly to insufficient rigor.

It is claimed that the inability of students to understand the utmost refinements of mathematical thinking may be overcome by making them study — if necessary, memorize — perfect models. By learning a great many models the student is said to acquire, in some unexplained manner, understanding of and ability to do mathematical work.*

The psychological method. — The followers of the psychological method assert that in any subject it is a mistake to consider only the scientific aspect of the subject, and to ignore absolutely the degree of mental development of the student. An exactness which is not understood by the student is not exactness so far as he is concerned. Over-rigorous teaching will not lead to

* The assertion that mathematics can be studied by reading and rereading perfect models, without any original thinking on the part of the student, can be made only by people who either are not capable of analyzing their own ideas, or who cannot do any mathematical thinking of their own. For otherwise they would notice the inefficiency of an occasional bit of study based upon reading alone. Such study does not lead the reader to observe the characteristic difficulties and the nature of the critical points of the subject, and hence the knowledge acquired in this manner is superficial and short-lived.

rigorous thinking. In most cases the result will be lack of interest, sometimes even disgust with a subject.

Rigor. — While exactness in thinking and accuracy in speaking are among the chief aims of mathematical teaching, exactness must never be carried to such an extreme as to make the subject unintelligible to the student. The objections to extreme rigor may be summarized as follows: *

1. There is no absolute rigor possible in a secondary school.

2. The rigor insisted upon in many classrooms is frequently only an exact adherence to a textbook, which may, and often does, contain flaws.

3. By studying exact models which he cannot understand, the student will not improve his reasoning power, for he does not do any *exact thinking*. He only repeats exactly somebody else's ideas.

4. Students lose interest in mathematics and acquire wrong notions of mathematics in general.

5. Students are frequently led to mechanical methods of study and to memorizing.

6. Students without mathematical ability, but with good memories, are made to consider themselves good mathematicians.

THE LECTURE METHOD AND THE HEURISTIC METHOD

The lecture method. — Scarcely any teacher in a secondary school would present geometry or algebra in the form of lectures. There are, however, teachers who

* Compare Chapter VI, Preliminary Propositions.

lecture too much, teachers who are inclined to state demonstrations or other pieces of work without questioning the students; in other words, teachers who present considerable portions of the work in the lecture form. Hence a brief discussion of the merits and drawbacks of the lecture method may be justified.

Merits. — 1. It allows the presentation of a large amount of subject matter within a given time.

2. It can be used for large audiences.

3. The logical sequence of ideas is not interrupted.

4. It is comparatively easy for the teacher.

Drawbacks. — 1. Receiving information is not mathematical study.

2. The attention of the students is likely to wander.

3. The ideas follow one another so rapidly that little is comprehended during the lecture, and a great deal is left to home study.

4. In mathematics, the inability to understand one essential point may make the rest of the lecture unintelligible.

5. The teacher is not in contact with his class, and is unable to determine whether or not the majority of the students are able to follow.

In general, lecturing, even to a small extent, is out of place in a secondary school, but is somewhat justified for advanced university work.

The heuristic method.* — The heuristic method at-

* The heuristic method is sometimes called the Socratic method, although the latter is only a special form of the former, viz. the leading *ad absurdum* by questions.

tempts to make students find and discover as much as possible, and to reduce direct information to a minimum. Since students of high school age are unable to make absolutely original discoveries, they must be led, and the heuristic method does the leading by questions.* Not only the teaching in the classroom but also the arrangement of textbooks may be based upon the heuristic plan.

Advantages. — 1. Pupils *think for themselves* and are not merely listening for information.

2. Students acquire a real understanding of the subject. A person listening to a lecture quite often does not grasp all the peculiarities of a demonstration; on his attempt a few days later to construct such proof, certain difficulties that were not noticed at the lecture interfere. A student, however, who discovers the solution of a problem himself has a full understanding of its difficulties and can easily reconstruct it.

3. The *interest* of the students and the resulting willingness to work are greater when they are taught heuristically, than when taught by informational methods. Mathematical instruction cannot be successful if it fails to stimulate the interest; and, on the other hand, interest is the most powerful stimulus for work. A student who is interested has no difficulty in paying attention, and is, as a rule, successful in his work. Mathematical study for one deeply interested is a

* When the questions are addressed to the entire class the members of which are expected to coöperate, when requested, the method has been called the "genetic." But as this is the mode of procedure of nearly all heuristic teaching the distinction is somewhat artificial.

pleasure, and not a drudgery. In general no truer criterion for the success of mathematical instruction exists than the interest of the students.

4. Teachers are in complete touch with their classes.

5. Home study is not nearly so heavy or tedious as when informational methods are used.

Disadvantages. — 1. The heuristic method is slow, especially in the beginning.

2. It is sometimes difficult to make students discover certain facts.

3. The method is difficult for the teacher, for he cannot simply follow a textbook, but must constantly seek devices for leading students, devices that must be modified for different pupils and for different classes.

4. The method does not work well in the hands of every teacher. Some teachers expect too much from the pupils, and consequently accomplish hardly anything. Others expect too little, making the questions so easy that students have to answer simply "yes" or "no." This is the fault of some textbooks that claim to be heuristic, and lead the students by questions like the following:

"Compare AB and AC."

"Compare $\angle BAD$ and $\angle DAC$."

"What is common to $\triangle$ ABD and ACD?"

"Compare $\triangle$ ABD and ACD."

"What conclusion can you draw with reference to $\angle B$ and $\angle C$?"

The ability to ask truly heuristic questions is most essential for the mathematical teacher.

THE LABORATORY METHOD

Description of the method. — The general dissatisfaction with the results of the prevailing dogmatic-informational methods has recently placed in the center of interest a method that has some similarity to the heuristic method. The *laboratory* method proposes also to lead students to the discovery of mathematical facts. The means of discovery, however, is not the questions of the teacher, but experiments performed by the pupil. By actual weighing and measuring, areas, volumes, lines, and angles are determined; and each particular mathematical relation is found as a consequence of a number of such experiments.

Thus, to make a student discover the relation between the area of a circle and its diameter, have him cut out of cardboard a number of circles, and let him determine the areas by experiment. These areas may, for instance, be found by comparing the weights of the cardboard disks with the weight of the unit of area, cut out of the same material. By making a list of diameters and weights, the student may find the relation:

$$\text{area} = 3\tfrac{1}{7}\,(\text{radius})^2.$$

It would lead too far to give extended lists of all the different schemes and devices proposed by the advocates of the laboratory method. The wide scope of the work, however, is indicated by the great variety of materials that have been used in such work, as drawing instruments, cross-section paper, measuring rods, calipers,

balances, transit instruments, sextants, steel tapes, thermometers, levers, pulleys, planimeters, etc.

Good features of the laboratory method. — The natural way of making discoveries, the way the human race has taken, is from the concrete to the abstract. Laboratory work is exceedingly concrete and hence interesting and enjoyable to young students. It emphasizes the *doing*, it requires the student to accomplish something that is within his capacity.

The laboratory method brings the applications of mathematics into prominence, while students taught otherwise are notoriously weak in applying their mathematics. Finally it gives the student a clear notion of the space concepts. A student who has measured many angles will naturally know what an angle is, a thing not at all certain in the case of students whose instruction was purely theoretic.

Weak features of the laboratory method. — 1. It is not at all easy to make students discover mathematical facts by experiment.

2. It is an exceedingly slow method.

3. It degenerates sometimes into a kind of manual training.

4. It is based upon the wrong assumption that pupils cannot comprehend, and do not enjoy, demonstrational mathematics.

5. Laboratory work and induction from experience are not typical mathematical work, and hence such methods used exclusively do not give the student any training in true mathematical thinking. It makes the

student acquainted with *mathematical facts*, but *not with mathematical reasoning*.

Summary. — Laboratory methods form an exceedingly valuable supplement to the teaching of mathematics. Students doing some work of this character will have more interest in, and understanding of, pure mathematics. But the laboratory method must not be pushed to the point of complete abandonment of pure mathematics.

CHAPTER IV

THE FOUNDATIONS OF MATHEMATICS *

THE AXIOMS OF GEOMETRY

The bases of geometry. — Every conclusion rests upon premises which either are self-evident, or which require demonstrations, *i.e.* lead to other premises. As this reference to more fundamental premises cannot be continued *ad infinitum*, every deductive science, and geometry in particular, rests upon a number of non-provable propositions, considered self-evident and called *axioms*. To define an axiom, however, as a self-evident truth would involve the assumption of an immutable standard of self-evidence. It will appear from the following section that axioms are sometimes merely conventions without much reference to their common-sense evidence; but in a preliminary sense we may say that an axiom is a proposition, (1) *assumed* as self-evident, (2) not capable of being deduced from other axioms.

Following Euclid, axioms relating to purely geometric facts, as "Two points determine a straight line," are frequently called postulates.† The entire subject of geom-

* It is impossible to give even an outline of this important subject within the available space. Here only a few fundamental phases of the subject, which seem to have philosophic or pedagogic importance, are discussed.

† The term "postulate" is used in elementary geometry in a somewhat different sense.

etry rests upon axioms, postulates, and definitions, and hence these are frequently called the *bases of geometry*.

The philosophic aspect of the geometric axioms. — The chief philosophic question in regard to mathematical axioms is: Does the knowledge of the mathematical axioms precede experience, or is it the result of experience?

The *rational* doctrine in general asserts that certain elements of reason must underlie all experience; that without them experience is impossible. In other words, the knowledge of certain facts must precede all experience; there is knowledge *a priori*. The *empirical* school, on the other hand, claims that all knowledge is finally derived from experience, and that there can be no knowledge *a priori*.

The mathematical axioms derive a special philosophic interest from the fact that they played a very important rôle in the controversy between these schools. The rationalists always pointed to geometry as an obvious example of knowledge independent of experience, — knowledge *a priori*. Geometry, they claimed, gives us some knowledge of the real world, and is independent of experience.

This argument can be met either by contending that geometry does not give us any knowledge of the real world, or by claiming that geometric axioms are experimental facts. While modern mathematics has made the first assertion very probable, most empiricists like Hume and Mill attempted to prove the second point. According to John Stuart Mill, the postulates

are neither exact nor necessary, and their apparent certainty is produced by the continuous presence of spatial impressions. Two diverging lines are always seen to diverge farther and farther, and this experience, often repeated and uncontradicted, is the reason why we cannot conceive the possibility of their meeting again. But this apparent certainty is only the result of induction, and does not prove that the opposite is impossible.

Kant,* on the other hand, considers the axioms as (synthetic) judgments *a priori*. He claims that they possess a universal and necessary certainty which no experience can give, and that the opposite is unthinkable. We cannot conceive the existence of a triangle the sum of whose angles is more than 180°.

Purely philosophical speculations have not advanced the solution of this problem as much as have the mathematical investigations which we shall briefly discuss in the next sections.

NON-EUCLIDEAN GEOMETRIES AND THE AXIOMS

Euclid's postulates. — Not considering statements that obviously can be proved, Euclid's geometry contains two geometric axioms, viz.:

1. Two points determine a straight line.

* Kant is not a rationalist in the strict sense of the word. He asserts: There is knowledge *a priori*, but this knowledge is only about things as they appear, not about things as they really are. Kant's rationalism is therefore *idealistic*, while before him it was *realistic*, *i.e.* relating to the knowledge of the real world.

2. Two intersecting lines cannot both be parallel to a third line.*

Tacitly assumed, however, is a third postulate, the so-called axiom of free mobility:

3. A figure can be moved from place to place without change of form.

The axiom of mobility is of great importance, for all proofs of equality and all measurements depend upon it, but up to comparatively recent times its tacit assumption was not recognized.

As the second axiom, frequently called *Euclid's postulate*, seemed to be capable of proof, innumerable attempts were made to deduce it from the other axioms, but without success. After centuries of continued failure this lack of proof was considered such a flaw in the apparently perfect structure of geometry that some mathematicians referred to it as the *defect* or *disgrace* of mathematics.

The geometry of Lobatschewsky. — Finally, however, a method first suggested by Gauss, and carried out by Bolyai and Lobatschewsky,† showed absolutely that Euclid's postulate cannot be proved.

This method, which may be called the indirect proof

* These axioms frequently appear in different forms; *e.g.:*

(1) (*a*) Two diverging lines cannot meet again.
(*b*) Two straight lines cannot inclose a space.

(2) (*a*) Two lines are parallel if two interior angles on the same side of a transversal are supplementary.
(*b*) The sum of the angles of a triangle equals 180°.

† Gauss communicated his results to Bolyai's father, 1795 to 1799, and Bolyai published his results in 1832. About the originality of Lobatschewsky's work, however, there is absolutely no doubt.

on a large scale, assumes that through a point several parallels to a line can be drawn, and, retaining the other axioms, draws a great many consequences, in fact builds up an entire geometry. If this assumption is wrong, — *i.e.* if Euclid's postulate is the necessary consequence of the other axioms, — then sooner or later contradictions must appear. The surprising result, however, is that the entire system does not lead to any contradictions. Lobatschewsky obtained an entirely new geometry, forming a perfect non-contradictory system that is as coherent and as logical as Euclid's.* This shows conclusively that Euclid's postulate is not a consequence of the other axioms, and that there is no *logical* reason against assuming that two intersecting lines may be parallel to a line.

Lobatschewsky's geometry was the first *non-Euclidean* geometry, but soon others were to appear.

The geometry of Riemann. — About twenty-five years later, Riemann constructed a geometry which is based upon a much deeper analysis of the axioms and the nature of space.† By retaining only the axiom of mobility, but rejecting the two other postulates, Rie-

* The theorems of this geometry differ in many cases from those of Euclid, and some appear at first rather singular, *e.g.*:

1. The sum of the angles of a triangle is always less than 180°.
2. The area of a triangle is proportional to the difference between the sum of its angles and 180°.
3. Two perpendiculars erected at two points of a line diverge.
4. There are no similar figures.
5. Space is finite, but unbounded, etc.

† Riemann, Ueber die Hypothesen die der Geometrie zu Grunde liegen.

mann obtains results that show a striking analogy to those of Lobatschewsky, being in some cases identical, in others opposite.* Riemann's two-dimensional geometry is identical with Euclid's spherical geometry, while his solid geometry involves curved space, a notion based upon certain investigations of Gauss.†

While Lobatschewsky's geometry did not lead to any

* *E.g.*, 1. The sum of the angles of a triangle is always greater than 180°.

2. The area of a triangle is proportional to the sum of its angles diminished by 180°.

3. Two perpendiculars erected at two different points on a line converge.

4. There are no similar figures.

5. Space is finite, but unbounded, etc.

† It is frequently attempted to make the notion of curved space somewhat more plausible by the following considerations: Imagine (if you can) two-dimensional beings who live on the surface of a huge sphere, and whose physical constitution does not allow them to perceive anything outside this spherical surface. As long as such beings are acquainted only with a small portion of the sphere, they will consider their world a plane; and if they construct any geometry, it will be the plane Euclidean geometry. If they should, however, become acquainted with a comparatively large portion of this sphere, they would recognize that their world is a sphere, and that their former geometry was not exactly true. They would construct a spherical geometry and recognize that what they considered straight lines are really circles; that true straight lines cannot exist; that their world is finite and unbounded, etc.

Similarly our notions of three-dimensional space may be due to limited experience, and an acquaintance with larger portions of the world — so it is argued — may show that space is curved, a notion which we cannot imagine, but whose consequences we can draw. In such a space, apparent planes would be spherical surfaces, lines would be circles, true straight lines and planes could not exist, such a space would be finite and unbounded.

This whole explanation is, however, so full of difficulties and impossibilities that it has no scientific value.

contradictions, there was the possibility that by further development hidden contradictions might be revealed. Riemann's two-dimensional geometry, being identical with the Euclidean spherical geometry, cannot possibly lead to any contradiction. As the same conclusion can be generally proved for his solid geometry, Riemann's work was not open to the only objection that could be made to Lobatschewsky's.* Thus we have three geometries, all three equally true as far as logic can decide, although not all three equally convenient for practical purposes.

Modern view of geometric axioms. — The fact that different parallel axioms produce perfect logical systems

* Soon afterward, however, Beltrami showed that Lobatschewsky's two-dimensional geometry is identical with the geometry of figures that can be drawn on the so-called surfaces of constant negative curvature, and removed thereby the above-mentioned objection.

A surface of constant negative curvature is either a pseudosphere or a surface that may be deformed so as to be applied to a pseudosphere. Similarly, a surface of constant positive curvature is a sphere or any of its deformations, and a surface of zero curvature is a plane or any developable surface (as cone, cylinder, etc.).

The different two-dimensional geometries may be represented in the following surface:

Geometry	Constant Curvature	Typical Surface
Riemann	+	Sphere
Euclid	o	Plane
Lobatschewsky	—	Pseudosphere

The mathematical expression for measurement of curvature is $\frac{1}{R_1R_2}$ where R_1 and R_2 are the principal radii of normal curvature.

proves that we have no logical reason for adopting Euclid's postulate. Reasoning, unsupported by experience, would be absolutely unable to make a choice between the three parallel axioms; and hence the view of the rationalists that the postulates are *a priori* facts, the knowledge of which precedes all experience, and the opposite of which is unthinkable, is hardly tenable. In regard to the true nature of the axioms we may say that pure geometry selects its axioms in the same way that it chooses its definitions, viz. by accepting conventions that do not lead to contradictions, without regarding in the least our physical experiences. Most mathematicians therefore consider geometric axioms as conventions.*

The general acceptance of the Euclidean axiom in preference to non-Euclidean axioms is due to experience. Hence, we may say that Euclidean geometry is partly based upon experience or induction.

Summary of the consequences of non Euclidean geometry. — The view is not unfrequently expressed that non-Euclidean geometry — or Metageometry as it is sometimes called — is a logical curiosity opposed to all common sense, and without any mathematical or other value. As far as any knowledge of the real world is concerned this may be true, but the study of these subjects has nevertheless distinct advantages, some of which are the following:

1. It positively demonstrates that Euclid's postulates

* This does not of course relate to the general axioms which have no geometric character.

cannot be derived from the other axioms. Before Lobatschewsky mathematicians wasted a great deal of effort in the vain attempt to prove the impossible, while at present the problem belongs in the same class as the squaring of the circle, or perpetual motion.

2. The knowledge of the foundations of geometry, and especially the true nature of the axioms, has been put on a much more scientific basis than before.

3. The argument of the rationalists that geometry gives us an *a priori* knowledge of the world has been met. This has been accomplished, however, not by following Mill's line of argument, but by proving that *pure* geometry cannot give us any knowledge of the real world.

Other geometries. — Although some geometers were inclined to consider the three geometries as the only possible ones, it was soon recognized that other axioms had been tacitly assumed whose denial led to other systems. The main tendency of this development * has been to put Euclidean as well as non-Euclidean geometries upon a stricter logical basis by freeing it from all elements of sense-perception. Things which we perceive to be obviously true by inspection of the diagram cannot be admitted without argument, for seeing is not demonstrating. Thus, if four collinear points, A, B, C, D, are so located that C lies between A and D, and B between A and C, then we can see that B lies between A and D. Such an argument, however, could not be admitted in the rigorous geometry. It has either to be

* The best-known investigator in this field is possibly Hilbert. See Hilbert, The Foundations of Geometry, Chicago, 1902.

demonstrated, or if consistent with the other axioms, may be admitted as an axiom.

Thus we obtain highly logical geometries, built upon numerous axioms, and free from empirical elements. The figures to which they refer, however, have little in common with the figures with which our senses have made us familiar; and there is no rigorous proof that the results of such geometries agree with the results of the geometry that refers to real spatial objects. In other words, these systems, while highly perfect from the point of view of logic, are not applicable — at least not if we wish to retain our high standard of rigor. We may almost speak of systems of logical symbols that have no meaning and that do not lead to any applications.

THE FUNDAMENTAL LAWS OF ALGEBRA

The laws. — The letters used in elementary algebra always represent numbers,* and hence this science may be considered an extension of arithmetic, and its fundamental laws must be identical with those of arithmetic.† It is, however, not easy to recognize that certain laws are tacitly assumed in all arithmetical operations, and men added and subtracted for many centuries without

* There exist, however, other, non-numerical algebras.

† A complete outline of this matter cannot of course be given in a few pages, hence only a few illustrations are given above. The reader is referred to Encyklopädie der Mathematischen Wissenschaften, Band I, Schubert, Grundlagen der Arithmetik, Leipzig, or to the French edition of the same work: Encyclopédie des sciences mathématiques pures et appliquées, Gauthier Villar, Paris.

recognizing these laws, until in the first half of the nineteenth century the investigations of English and French mathematicians revealed their existence.

To show the assumption of some of these laws let us consider a simple example of addition, *e.g.* $17 + 8 = 25$. The number 25 is a symbol for $20 + 5$; hence our example is $17 + 8 = 20 + 5$. To obtain the 20 we must obviously divide 8 into two parts, 3 and 5, and add 3 to 17. Or, we have:

$$17 + 8 = 17 + (3 + 5) = (17 + 3) + 5 = 20 + 5;$$

i.e. we have assumed the associative law:

$$a + (b + c) = (a + b) + c.$$

Similarly, to multiply 8 by 13 we have:

$$8 \times (10 + 3) = 8 \times 10 + 8 \times 3; \quad \text{(Distributive Law)}$$

or, since we multiply units first:

$$\begin{aligned} 8 \times 10 + 8 \times 3 &= 8 \times 3 + 8 \times 10 && \text{(Commutative Law)} \\ &= 24 + 80 \\ &= (4 + 20) + 80 \\ &= 4 + (20 + 80) && \text{(Associative Law)} \\ &= 4 + 100 \\ &= 100 + 4. && \text{(Commutative Law)} \end{aligned}$$

Some of the most important laws for addition are:

The commutative law, $\quad a + b = b + a.$

The associative law, $\quad a + (b + c) = (a + b) + c.$

For multiplication:

The commutative law, $ab = ba$.

The associative law, $(ab)c = a(bc)$.

The distributive law, $a(b + c) = ab + ac$.*

All arithmetical calculations consist of applications of these laws and of the multiplication and addition tables for units. Since these laws underlie all algebra and ultimately all mathematics, various attempts have been made to account for them. Some mathematicians have attempted to derive some of these laws from the remaining ones or from other still simpler ones.† Others tried to derive them from more general concepts.‡

But all such investigations leave a remainder, for the explanation of which two theories have been elaborated, viz.:

1. The realistic view.
2. The formal view.

The realistic view. — According to the realistic view mathematics refers to, and is capable of giving us information about, real things. Mathematics is partly based upon experience, and numbers represent primarily numbers of things. Our knowledge of the fundamental laws of algebra is due to sense-perception.

* The above illustrations are sufficient for our purpose, but there are other laws, *e.g.* the results of addition and multiplication are one-valued; multiplication and addition are always possible.

† Peano, Arithmetices principia nova methodo exposita, Torino, 1889.

‡ *E.g.* theory of assemblages. See Dedekind, Was sind und was sollen die Zahlen, Braunschweig, 1888.

We can see that $2+3=3+2$, if we observe the annexed diagram:

$$\bullet\ \bullet \quad \bullet\ \bullet\ \bullet = \bullet\ \bullet\ \bullet \quad \bullet\ \bullet$$

Similarly, the next diagram shows that $2 \times 3 = 3 \times 2$:

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{array} \qquad\qquad \begin{array}{cc} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \end{array}$$

Of course we cannot see that $245 \times 727 = 727 \times 245$, but mathematical induction enables us to prove that these laws are generally true, if we can establish them for some numbers. Thus algebra is considered to have its roots partly in experience; it is to some extent an inductive science.

The formal view. — This *assumes* the fundamental laws as the basis of arithmetic and algebra. These laws are definitions, and algebra is what it is in consequence of these definitions, and not in consequence of any of its applications. Any particular arithmetic problem as $\frac{a}{b}+\frac{c}{b}$ or $\frac{3}{7}+\frac{2}{7}$ is not solved by reference to any real things (*i.e.* what we would obtain if we added $\frac{3}{7}$ and $\frac{2}{7}$ of a thing), but solely by the fundamental laws.

After the acceptance of the laws, the letters a, b, c, d, x, etc., are simply symbols which obey these laws, and their meaning is not in the least determined or restricted by any possible application; in fact, they do not mean anything.

Thus algebra, like rigorous geometry, becomes an abstract system of symbols that depends solely upon the arbitrarily assumed laws, that is independent of

experience, and that has no connection with the real world. This system would be perfect if the consistency of the fundamental laws could be proved, since inconsistent principles will lead to contradictions.* But this proof has not been given, and is not likely to be given.

Comparison of the two views. — The realistic view is not able to put every element of algebra upon a logical basis. It must admit that mathematics contains a few elements of empirical character. While this means the giving up of logical perfection, it produces on the other hand an algebra that is *applicable*.

The formal view of algebra places the subject upon a more logical basis, but its results cannot be directly applied. It would be necessary to demonstrate that this formal algebra is identical with the applicable algebra, a proof that in all probability cannot be given. For it is clear that a symbolism based upon arbitrarily assumed laws cannot give any information about real things.

But applications are the life of mathematics. Without applications it would never have been invented; without applications it would soon be forgotten. Hence if we could make algebra an absolutely exact logical structure, that is not applicable, it would lose its value. Nobody would care to study a system of symbols which mean nothing, which are connected by laws that mean nothing, and which lead to results that mean nothing.

Conclusion. — Thus the tendency of modern investigations has been to split mathematics — geometry as

* The realistic theory does not need such a demonstration, since operations with real things cannot lead to contradiction.

well as algebra — into two parts, viz. the applicable mathematics and the rigorous formal mathematics. The latter has been made exceedingly rigid and logical and has been freed from almost all elements of experience, but at the expense of its applicability.

Everything in this formal science is built upon arbitrary assumptions that are logical, but that have no relation to real things. Hilbert's planes have very little in common with real planes; formal numbers are not numbers of real things. While the rigorous formal part has undoubtedly done a great deal to clear up many philosophical questions relating to mathematics, it would have little interest if there were no applicable mathematics.

PEDAGOGIC CONCLUSIONS

None of the facts relating to the foundations of mathematics has a place in secondary schools. A student must become acquainted with quite a number of mathematical facts and theories before he can understand and appreciate investigations of so difficult a nature. Hence the attempts of some writers to present these matters to high school students cannot be recommended.

But these facts do influence the teaching of mathematics indirectly. The modern investigations of the foundations of mathematics prove clearly that the system of geometry as found in school textbooks is not the absolutely rigorous system that a few decades ago it was believed to be. Errors and assumptions have

been shown to enter into many elements of that supposedly flawless system of logic. A study of the foundations of mathematics will convince the teacher who insists upon a repetition of every word of his geometric gospel, that the perfect rigor exists only in his imagination.

The fundamental laws of algebra have no great value for beginners; but if they are mentioned at all, they should be explained by means of concrete arithmetical examples. In regard to the teaching of axioms in high schools, an excess of rigor is out of place. Thus we do not need to strive for absolute completeness when compiling lists of axioms. A statement may be considered an axiom even if it can be deduced from other axioms; and a common sense reason may sometimes be given when technically an axiom should be quoted.

CHAPTER V

DEFINITIONS

LOGICAL ASPECTS OF DEFINITIONS

What is a definition? — The number of ways of defining the term "definition" is as large as the number of treatises on logic and the number of dictionaries.* For the purposes of elementary mathematics, however, the old scholastic definition, although little used by modern writers on logic, is exceedingly useful, viz. *A definition is the designation of the proximate genus and the specific difference.* To define a term we must state the proximate genus, *i.e.* the nearest class to which it belongs, and the specific difference, *i.e.* the particularities that distinguish it from all others of the same genus.† Thus for a parallelopiped the proximate genus is "prism," and the specific difference is the fact that its base is a parallelo-

* Among the more widely known may be mentioned: "A definition is the explaining of a term by means of others, which are more easily understood" (De Morgan); "A concise account of the essential and characteristic properties of a thing;" "A description or an explanation of a word, thing, or symbol that distinguishes it from all others" (Standard Dictionary); etc.

† The view that "all definition is classification" (Erdman, Die Axiome der Mathematik) may not be generally true, but it is certainly useful in mathematics. Complete classification of a set of terms frequently facilitates the understanding of their meaning. Thus in defining quadrilaterals, it is advisable to represent a complete scheme of classification on the blackboard in which each kind of figure finds its proper place.

gram. Hence a parallelopiped is a prism whose base is a parallelogram.

Common errors. — 1. *Errors in genus.* The genus given in a definition must be the proximate, *i.e.* the nearest one. Hence it is wrong to define a pentagon as a plane figure bounded by five straight lines, for the nearest genus is the polygon. Similarly, in a definition, a parallelogram must not be classified as a polygon, nor a triangle as a portion of a plane, nor an octaedron as a solid, nor a parallelopiped as a polyedron, etc.

2. *Errors in the differentia. Redundancy.* Neither more nor fewer differences must be given than are necessary to determine precisely the meaning of the term. The statement, "An inscribed polygon is a polygon whose vertices lie in a circumference, and whose sides are chords," is redundant, since two differences are given, each of which is the consequence of the other. If the vertices lie in the circumference, the sides must necessarily be chords, and *vice versa.*

Redundancy, however, is particularly objectionable if the differences which are given are not obviously compatible. Such a definition either involves a theorem or it defines an impossible thing. "A parallelogram is a quadrilateral whose opposite sides are equal and parallel" involves a theorem; while the statement, "A spherical square is a spherical quadrilateral whose sides are equal and whose angles are right angles," gives differences that are incompatible, and hence refers to a figure that cannot exist. Certain time-honored redundancies, however, such as the usual definitions of rhom-

bus, rectangle, prism, perpendicularity of line and plane, are justifiable on pedagogic grounds, since they convey at once to the student a clear notion of the true shape of the figure, and the slight error made may be easily corrected when the corresponding theorem is studied. The definition, "A rectangle is a parallelogram whose angles are right angles," is redundant, but it gives a clearer mental image of the figure than does the correct definition.

It is doubtful whether the redundancies that are sometimes used in more advanced chapters of the geometry for the purpose of avoiding difficult proofs, can be justified. A regular polyedron is nearly always defined as "a polyedron whose faces are equal regular polygons, and whose polyedral angles are equal." The equality of the polyedral angles follows from the fact that all faces are equal regular polygons.

Difficulties inherent in certain definitions. — Two classes of terms are most difficult to define, viz. general class names, as mathematics, geometry, algebra, functions, etc., and very fundamental terms, as line, direction, plane, etc. The former can be formulated and appreciated only by advanced students who really know what these terms mean; and their importance for secondary schools is consequently very small. The latter we shall consider at somewhat greater length.

Just as all theorems rest finally upon a number of supposedly self-evident propositions or axioms, so all definitions must ultimately be based upon a few exceedingly simple ones. Hence there exist a number

of terms, such as space, boundary, position, direction, straight line, plane, etc., which are either not capable of definition or which are hard to define from the "difficulty of finding ideas more simple and intelligible than the ones to be defined." Most of these are accepted without definitions; a few, however, are defined in nearly all textbooks and are taught in most schools.

Line. — A typical example of this kind is represented by the various definitions of a straight line which appear in the widely used textbooks, all of which upon closer analysis prove to be faulty.* Thus the commonly given definition, "A line that has the same direction throughout its length," involves the term direction, which is not simpler or more self-evident than straight line. The statement, "A straight line is the shortest distance between two points," is not a definition, but a theorem, and makes use of distance, a term based upon straight line. The definition, "A straight line is a line determined by two points," is possibly the most absurd of all of them, for it is utterly unintelligible to the beginner, who does not know the technical meaning of the phrase "determined by," and who, if the phrase should have for him any meaning at all, may think of other lines, *e.g.* the minimum circle, which is also "determined" by two points. Similarly all other definitions contain flaws.

* It is often claimed that the well-known definition given by Gauss is absolutely exact. But aside from the fact that scientific objections have been made, it is one that has no value for secondary school work. It is based upon the following consideration: If a figure moves (rotates) while two points in a line remain fixed, the line is a straight one if all its points are motionless while all the rest of the figure moves.

They relate to motion and involve the notion of time, or are vague and almost meaningless statements.* There exists no flawless definition of straight line which is fit for school use, and undoubtedly the best policy would be to accept this term without definition.

Angle. — Similar difficulties are encountered in the formulation of a definition of angle. Usually the word "angle" is defined by using terms like inclination, direction, rotation, etc., whose meaning is either not clearer than the meaning of angle, or which imply mechanical notions. Undoubtedly the idea of rotation furnishes a splendid *explanation* of what an angle is, but not a formal definition. Often an angle is defined as the space between two lines. An infinite space, however, is a most peculiar idea for a beginner, and the comparison of several infinite spaces, *e.g.* the statement that one infinity is twice or thrice another infinity, certainly will lead to difficulties that are beyond the student's grasp. Moreover, an angle is not a space (it is not measured in square units), but only the (partial) boundary of such a space.

The incompleteness or faultiness of many definitions of angles usually becomes apparent when they are applied to straight angles, reflex angles, or angles greater than 360°. This test shows the limitations of the definitions that are based upon inclination, divergence, direction, etc.†

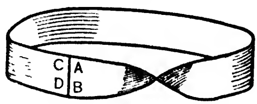
* *E.g.* a straight line is the simplest line possible.

† The widely advertised definition, "An angle is the figure formed by two straight lines diverging from a point," has the same drawback. Moreover, it is most unpedagogical for the reason discussed on page 74.

Plane. — Similarly no definition of the term plane exists that is free from objections, and the endeavors of some of the greatest mathematicians to formulate such a definition have proved to be futile. The commonly used definition of a plane is a theorem, and the only way to justify its acceptance would be to consider it as a geometric axiom or postulate.

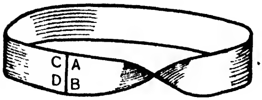
"A plane is the locus of a point equidistant from two fixed points" involves "locus," a term that is studied much later. "The sum total of all perpendiculars that can be drawn to a given line at a given point" (Fourier) makes use of the term perpendicular, which, in turn, needs plane for its explanation. "A surface that in its entire extent has only two dimensions" is vague and uses dimension. "If a straight line passing through a fixed point slides on another straight line, it generates a plane" uses mechanical ideas and leads to difficulties when the moving and the fixed line meet at infinity, etc., There exists no exact definition of the term plane.

Surface. — A surface is usually defined as the boundary of a solid. No doubt certain surfaces are boundaries of solids, but there are others which cannot be considered such, as for instance the surface represented in the annexed diagram * (Möbius leaf). It would be impossible to make



* This surface is obtained by joining AB and CD, the opposite sides of a rectangle, so that the lower end of each line touches the upper end of the other.

this surface the boundary of a solid, as becomes apparent if we attempt to color that side of the surface which is in contact with the solid, while leaving the other side white. The surface has only one side.



Hence if we attempt to be rigorous, we cannot base the definition of surface upon the boundary notion, but mu consider it the limit of a solid of vanishing thickness — a notion that cannot be considered for school work.

PEDAGOGIC ASPECT OF THE DEFINITION

Pedagogic value of formal definitions. — Indefinite and variable use of technical terms has been a source of errors and misunderstandings in many sciences, especially those of philosophic character, and consequently the absolute necessity of laying down exact definitions has frequently been pointed out. In elementary mathematics, however, the terms used are so definite that misunderstandings about their meaning will scarcely ever arise on account of lack of good formal definitions. The difference between a straight line and a curved line, between a binomial and a trinomial, between a trapezoid and a parallelogram, can be fully comprehended by persons who are not familiar with the formal definitions of these terms.

The great emphasis put upon the teaching of formal definitions in secondary schools is therefore usually defended, not on account of the importance of knowing these definitions, but on grounds of the "logical train-

ing" it is said to give. While it must be admitted that this study could be made such a training in logic, it is unfortunately very seldom developed in such a way in our schools. In a large number of cases, *the giving of a definition* is simply the verbatim repetition of a number of words. To be a logical exercise, it would be necessary to make the students themselves formulate these definitions. Even assuming that the average student were able to do this, it is evident that he could do so only *after* he had acquired a clear notion of the thing to be defined. Formal definitions would then form the end, and not the starting point of the study of a term. They would be studied, not for the purpose of conveying to the students the meaning of a word or thing, but merely for the practice in formal logic which they afford.

Such work would undoubtedly have a certain value; but we should not forget that not all definitions are fit for such exercises, and that the matter is really somewhat foreign to and not necessary for the study of elementary geometry. Hence the study of formal definitions should not be overdone. To judge from the prominence given to this topic by most examination papers set by examination boards and colleges, it seems that the value of formal definitions is greatly overrated.

A definition is not necessarily an explanation. — A logical definition is a description that distinguishes a thing from all others. It enables us to recognize and to identify a certain thing as such, but it is obvious that such an identification does not necessarily explain the true nature and the real character of the thing. We

may accept the following statement as a definition of an angle: "An angle is a figure formed by two rays diverging from a point." This definition may enable us to recognize an angle as such, but it would not give us the slightest notion of what really constitutes an angle. It would not enable us to apply this concept to further work. Formal definitions are often purely external, leading to identification of a thing, but not to an understanding of its true character.

Taking this fact in connection with the firmly acquired habit of students of accepting and repeating words without understanding their meaning, it is not surprising that students may know a formal definition of a word, without having a clear notion of its meaning. Hence, *explanations* of terms are really more important than definitions, and every new term should be fully explained and its meaning illustrated by concrete examples.

Familiarity with technical terms one of the most essential prerequisites for effective study. — Small as the value of *formal definitions* is, it must not be inferred therefrom that the study of terms in general may be neglected. On the contrary it is most essential for further study, and a number of students who can reason logically fail because they are slow in acquiring a full and clear understanding of the space-concepts. Others again have difficulties in retaining such notions even if they originally understood them. It is a common experience to find in the upper classes students whose notions of the term "angle" are — to say the least — hazy. Such an able mathematician as Professor

Minchin narrates that he studied six books of Euclid without knowing the real meaning of the word "angle."

It is a very easy matter to make every student in a class understand the meaning of the term "similar polygon"; and yet, a few weeks after, some students in the same class will believe that mutually equiangular polygons are similar.

A *single explanation* of a term will *rarely produce* that *complete familiarity* with terms that is absolutely necessary for placing a student in the most favorable position for attacking demonstration work. To have only "one difficulty at a time," terms must be made so familiar to the student that he can recognize the nature of a diagram in *any* position *without any mental effort*, and that the names call up automatically the proper mental images.

The methods for obtaining such familiarity with technical terms consist principally in the solution of simple exercises referring to these terms. A few illustrations of such methods will be given in the following sections.

THE TEACHING OF THE INTRODUCTORY DEFINITIONS

General remarks. — The attitude of the student toward any new subject is usually one that ought to make the introductory work in geometry comparatively easy. He brings with him a considerable amount of curiosity that produces interest. He is not yet biased against the subject by previous bad experiences. Do not kill off the natural interest by emphasizing words, and making the student repeat phrases which he does not under-

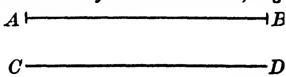
stand. Do not insist upon a verbatim recitation of every definition. Do not dwell too long upon difficult terms that are used very little. Use the heuristic method as far as possible; make the recitation lively and interesting, and try to avoid the impression that geometry is a study of words.

The difficulties which were discussed in the preceding chapter are very pronounced in the study of the introductory definitions, and a bad start might unfit the student for further work.

Surfaces and lines. — It would be a mistake to mention in a secondary school the difficulties that are inherent in such definitions as surface, plane, line, etc. To the beginner a surface is a boundary, as the boundary between a window pane and air, and a line is a boundary of a surface. In particular, point out that in the diagram of a line (as the annexed one) not the black line *AB*, but the boundary between white and black is a geometric line; that there is a line above the black, and another one below the black.

Do not define straight line, but assume it as an ultimate term. Point out the distinction between a line of definite and a line of indefinite length.*

* Unfortunately the English language has no generally accepted term for a line of definite length, and the terms used by some authors, *e.g.* segment, have serious objections. The writer uses a graphic way of pointing out this difference. Lines of definite length have ends marked by little cross marks, as *AB*; while lines of indefinite length have no such marks, as *CD*.



Angle. — *Explain* an angle as a rotation, by using a material contrivance that actually shows a rotation of a line. A pair of compasses, or, better, several pairs of different sizes, the hands of a toy watch, or even a book may be used effectively to show that an angle is generated by a rotation, and that the amount of rotation, and hence the angle, does not depend upon the length of the sides. Such illustrations will show clearly what an angle really is, and explain the meaning of straight angle, reflex angle, etc.

Methods of familiarizing the student with the notion of angle and related terms. — One of the most essential prerequisites for further study is the student's familiarity with technical terms. As pointed out in the preceding section, such familiarity cannot be obtained by the study of words, but by actual work involving these terms. The simplest exercises would consist (*a*) in having the student draw the various figures, as an acute angle, obtuse angle, adjacent angle, etc., and (*b*) in requesting him to name various figures which are drawn at the board. Then should follow exercises of more complex character. Three classes of exercises are available at this stage of the work, viz.:

1. Numerical Exercises.
2. Drawing Exercises.
3. Laboratory Exercises.

1. Numerical exercises involving the term "angle" and related terms.* — Most exercises of the following list

* The "originals" found in this book are given principally to assist the teacher in framing for himself questions of this kind. Hence such

should be done quickly and orally; the teacher drawing the diagram at the blackboard and assigning numerical values at random. A teacher who has had no previous experience should make himself quite familiar with such questions in order to be able to extemporize such problems whenever necessary.

A few of the more difficult problems may be assigned for written home work. It is, of course, usually not necessary to solve every exercise of the following set; and on the other hand, if more should be needed, it is a very simple matter to enlarge the list.

*a. Exercises to familiarize the student with the nature of the angle.**

(Request answers in various units, as, degrees, right angles, straight angles.)

Ex. 1. Find the angles formed by the hands of a clock at 1 P.M., 4 P.M., 6.30, etc.

Ex. 2. What angle is formed by lines drawn towards north and northeast; towards S. and S.E.; towards N.W. and S.W.; towards N.N.E. and N., etc.?

Ex. 3. Over what angle does the large hand of a watch sweep in 10 min., 15 min., 30 min., 45 min., 1 hr., etc.?

Ex. 4. Over an angle of how many degrees does a spoke of a wheel sweep when the wheel makes $\frac{1}{4}$ of a revolution, $\frac{1}{8}$ of a revolution, 2 revolutions, etc.?

Ex. 5. How large is each angle at the center if a pie is divided into 5, 6, 8, etc., equal parts?

exercises only are given as relate to work generally neglected in text-books, or such as the teacher is frequently called upon to extemporize in the class.

* In order not to increase the bulk of this work unduly, many exercises are stated with the utmost brevity.

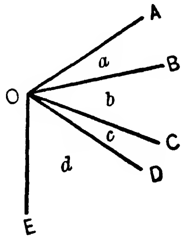
b. Addition and subtraction of angles; adjacent angles.

Ex. 6. Read by three letters: $\angle a, b, b+c, c+d, a+b+c$, etc.

Ex. 7. Which angles are adjacent to $\angle BOC$, to $\angle COD$, to $\angle AOB$, to $\angle DOE$?

Ex. 8.

Assign numerical and literal values to the angles in this column; as $40°$, $n°$, $\frac{1}{2}$ rt. $\angle$, etc.	Require the numerical value of angles in this column.
a, b	AOC
b, c	BOD
a, b, c	AOD
AOC, a	b
AOD, a, c	b
AOE, a, b, d	c
AOE, a, d	BOD
AOC, BOD, b	AOD
AOD, c, COE	AOE

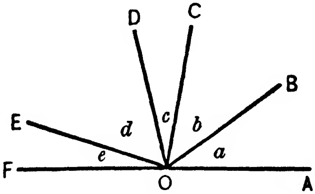


Ex. 9. In a similar diagram $AOE = 120°$ and $a = b = c = d$. Find a.

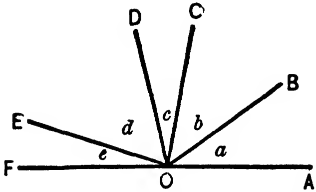
Ex. 10. In a similar diagram $a = b = c = 2d$ and $\angle BOD = 80°$. Find $\angle d$, etc.

c. Supplementary angles and straight angles.

Ex. 11. Find the supplement of $60°$, $30°$, $10°$, $\frac{1}{2}$ rt. $\angle$, $m°$, $(m+n)°$.



Ex. 12. In the annexed diagram, if AOF is a straight line, which angle is the supplement of $\angle a$, of $\angle AOD$, of $\angle(a+b)$, of $\angle(a+b+c+d)$, etc.?



Ex. 13. In a diagram similar to the preceding:

Assign numerical or literal values to the angles in this column:	Require the value of the angles below:
a, b, c, d	e
a, b, c, e	d
AOD, d	e
AOC, c, e	d
AOC, b, BOD	FOD
Or let $a = b = c = d = e$, and require	a
$a = b = c = d = 2e$, " "	e
etc.	etc.

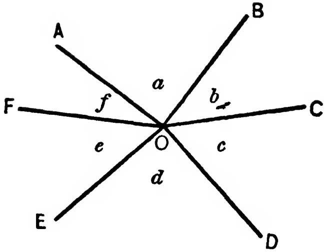
d. Complementary angles and right angles lead to questions quite similar to the preceding set. (Make $\angle AOF = 90°$.)

e. All angles formed at a point.

Ex. 14.

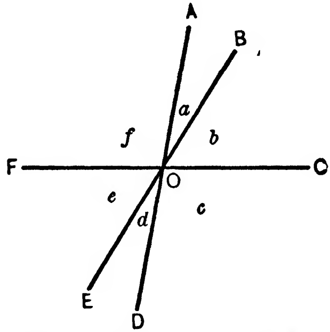
Assign values to	Require values of
a, b, c, d, e	f
a, b	Reflex $\angle AOC$
a, b, c, d	Reflex $\angle AOE$
Reflex $\angle AOC, b$	$\angle a$
AOC, b, BOD, DOF	$\angle f$
etc.	

Ex. 15. In a diagram similar to the preceding one, find a, if $a = b = c = d = e$ and $f = 20°$. Find b, if $a = b = c = d$, and $\angle AOF = 40°$, etc.



f. To practice the term "vertical angles," and to make some more difficult calculations. (The theorem of vertical angles should not be assumed for these questions.)

Ex. 16. If AD, BE, and CF are straight lines meeting at O,



find the vertical angles of each of the following angles: a, c, BOD, COE, $a + b$, $e + d$.

Ex. 17.

Assign values	Require values
a	AOE, d, BOD
AOC	c, DOF, f
a, b	c, d, e, f
a, c	b, d, e
f, d	b
AOC, COE	BOD

(Similarly for 4 or 5 intersecting lines.)

g. To become familiar with the term "bisecting."

Ex. 18. Draw the bisectors of a pair of complementary adjacent angles, assign various numerical values to one of these two angles, and require the angle formed by the bisectors.

Ex. 19. From the result of Ex. 18 find (by induction) a general truth about the angles formed by the bisectors of a pair of complementary adjacent angles.

Ex. 20. By assigning a literal value (*e.g.* m°) to one of a pair of adjacent complementary angles, *prove* that their bisectors always include an angle of 45°.

Ex. 21. Form three exercises corresponding to Exs. 18, 19, and 20 for a pair of supplementary adjacent angles.

Ex. 22. Form three exercises corresponding to Exs. 18, 19, and 20 for the bisector of an acute angle and the bisector of the reflex angle formed by its sides.

Ex. 23. Form similar exercises for the bisectors of a pair of vertical angles.

Ex. 24. Draw two right angles ABD and CBE, each adjacent to an angle ABC. Assign numerical values to $\angle ABC$ and require the value of DBE.

Ex. 25. Derive (by induction) from the preceding exercise a general relation between $\angle s\ ABC$ and DBE.

Ex. 26. Prove the preceding result by making $\angle ABC = m^\circ$.

Ex. 27. Modify Exs. 24, 25, 26, by drawing one or both perpendiculars in a direction opposite to the one assumed in Ex. 24.

2. **Drawing exercises.** — Although students at this stage of the work are not acquainted with any problems of construction, it is an easy matter to show them, without proof, some simple constructions upon which drawing exercises may be based. The following exercises are based upon two constructions, (*a*) the bisection of an angle, and (*b*) the construction of an angle equal to a given angle.

Ex. 28. Bisect an acute angle, an obtuse angle, a reflex angle, a straight angle.

Ex. 29. Divide a given angle into (*a*) 4, (*b*) 8 equal parts.

Ex. 30. Construct the supplement of a given $\angle A$.

Ex. 31. Construct one half the supplement of $\angle A$.

Ex. 32. Construct a right angle.

Ex. 33. Construct a perpendicular to a given line at a given point.

Ex. 34. Construct the complement of a given acute angle.

Ex. 35. Construct one half the complement of a given acute angle.

Ex. 36. Construct the supplement of the complement of a given acute angle.

Ex. 37. Construct the complement of the supplement of a given obtuse angle.

Ex. 38. If A and B are given angles, draw $\angle(A + B)$, $\angle 2A$, $\angle 3A$, $\angle(180 - A)$, $\angle(90 + B)$, $\angle\frac{A}{2}$, $\angle\left(180 - \frac{A}{2}\right)$, $\angle\left(90 + \frac{B}{2}\right)$.

Ex. 39. Draw angles of $90°$, $45°$, $22° 30'$, $135°$, $270°$, $225°$, $67° 30'$, etc.

Ex. 40. If A and B are given angles, draw angles: $45° + A$, $\frac{A}{2} + B$, $\frac{A}{2} + \frac{B}{2}$, $\frac{A + B}{2}$, $2A + \frac{B}{2}$, $3A + 45°$, $4B - 90°$, $2A - 3B$, etc.

Ex. 41. Draw the supplement of $2A$, of $\frac{A}{2}$, of $A + B$, of $\frac{A}{2} + \frac{B}{2}$, etc.

Ex. 42. Draw the complement of $\frac{A}{2}$, of $\frac{B}{4}$, of $90 - \frac{A}{2}$, etc.

Ex. 43. Draw the vertical angles of the following angles: A, $\frac{A}{2}$, $A + B$, $\frac{A}{2} + \frac{B}{2}$, $180° - A$, $360° - 3A$, etc.

3. **Laboratory exercises.** — Hardly any other work impresses upon a student the true meaning of an angle as well as the actual measurement by means of a protractor or transit instrument. Since the cost of a protractor is so small as to make its use possible in every school, this

instrument only will be discussed. In using a protractor, however, the teacher cannot be too emphatic in pointing out that work done with this instrument is not true geometric work, but merely drawing. Otherwise students will not see the necessity of using ruler and compasses for constructing an angle of 135°, etc., when a protractor gives the result so much more easily. In general, the protractor may be used for two classes of exercises, viz. those that require:

(*a*) the construction of some angle.

(*b*) the measurement of angles.

Ex. 44. Construct an angle of 20°, 75°, 88°, 145°, 170°, 250°, 280°, etc.

Ex. 45. Draw a triangle having one side equal to 1 inch and the two adjacent (nearest) angles equal to 40° and 70°.

Ex. 46. Draw a quadrilateral $ABCD$ having $AB = 1$ inch, $\angle B = 80°$, $BC = 1\frac{1}{2}$ inches, $\angle C = 100°$, $CD = 1$ inch.

Ex. 47. Draw a quadrilateral $ABCD$ having $A = 70°$, $AB = 1$ inch, $\angle = B\ 100°$, $BC = 2$ inches, $\angle C = 120°$.

Ex. 48. Draw $\triangle ABC$ having $AB = 2$ inches, $\angle A = 50°$, $\angle B = 70°$, and measure $\angle C$.

Ex. 49. In triangle ABC measure $\angle C$ if $\angle A = 50°$, $\angle B = 70°$, and $AB =$ (*a*) 1 inch, (*b*) $1\frac{1}{2}$ inches, (*c*) $2\frac{1}{2}$ inches.

Ex. 50. Draw a quadrilateral $ABCD$ having $A = 100°$, $B = 90°$, $C = 75°$, $AB = 2$ inches, $BC = 1$ inch, and measure $\angle D$.

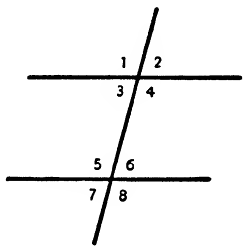
Ex. 51. Try to get some general truths from Exs. 48 and 49 (by induction).

It is possible to solve by drawing many problems that are usually given in trigonometry, such as the finding of heights, widths of rivers, etc., although such a detailed

statement would rather belong to a course in concrete geometry than to a course in demonstrative geometry.*

FURTHER ILLUSTRATIONS OF TEACHING DEFINITIONS

Angles formed by parallel lines and a transversal. — In studying the annexed diagram, the student should of course be able to recognize quickly the relations of such pairs of angles as 1 and 5, 3 and 6, etc. He should also be able to name quickly the other angle, if one of a pair of corresponding, or of a pair of alternate-interior, angles, etc., is mentioned.

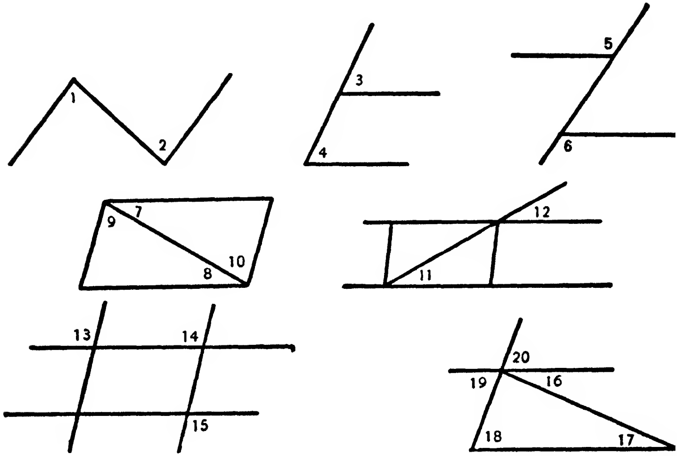


But to stop here would be insufficient. He should be able to recognize angles of the same kinds in diagrams that are incomplete, or that are complicated by additional lines, special emphasis being placed upon diagrams that occur later. Thus a student should recognize

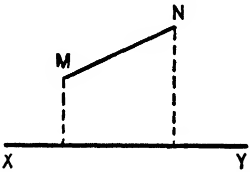
* In some school systems *concrete geometry* is taught in the grammar schools as a preparation for the demonstrative courses in the high school. One of the chief functions of such courses should be the thorough familiarization of students with the geometric concepts. Such a course should consist of "doing" and not learning facts. Drawing, counting, the making of plane and solid models, paper folding, graphic methods, use of cross-section paper, etc., should make the student thoroughly familiar with geometric terms, and give him a store of ideas to draw from in demonstrative work.

Unfortunately, however, a measurable increase of the student's mathematical knowledge is sometimes chiefly aimed at, with the resulting dislike of the student for the subject.

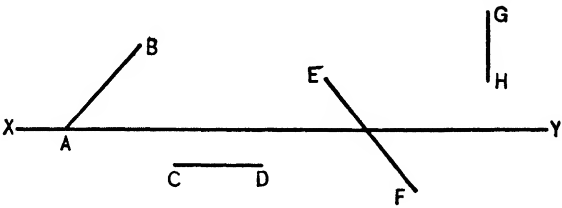
in the following diagrams the character of angles 1 and 2, 3 and 4, 5 and 6, etc.



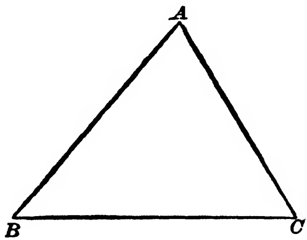
Projections. — The mere reference to the annexed diagram is not sufficient. The student should be able



to construct projections of various lines, as, *e.g.*, *AB*, *CD*, *EF*, *GH*, upon line *XY*. He should construct the



projections of various sides of an acute triangle ABC upon the other sides, as AB upon AC, BC upon AC,



AB upon BC, etc., and construct similar figures in an obtuse triangle.

CHAPTER VI

THE FIRST PROPOSITIONS IN GEOMETRY

PECULIARITIES OF THE PRELIMINARY PROPOSITIONS

The preliminary propositions. — The most fundamental propositions in geometry, such as "straight angles are equal" or "the complements of equal angles are equal," are frequently designated preliminary propositions. These preliminary propositions have certain peculiarities which make them less adapted to produce an understanding of geometry than are the theorems that follow.

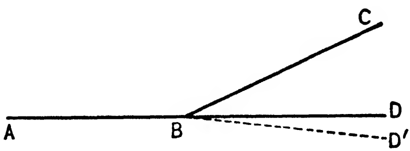
In the first place, these propositions state facts which are so self-evident that the beginner does not see the *necessity* of proving them. That right angles are equal, or that only one perpendicular to a given line can be drawn at a given point, are facts so obvious that their certainty does not appear to become greater by demonstrations of any sort.

In the second place, proofs of exceedingly simple facts are often difficult, and hence it is not surprising that many of the demonstrations given for the preliminary propositions are not the same simple deductions that are usually employed in geometry, but rather artificial devices. To the beginner such proofs frequently appear as unintelligible, complicated statements,

the truth of which is far more doubtful than that of the theorems to be proved.

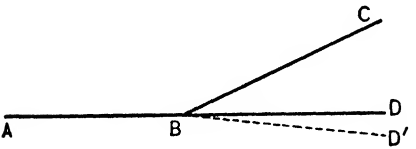
Usual mode of presentation. — Although absolute rigor is utterly unattainable when presenting this subject in a secondary school, many textbooks sacrifice pedagogic considerations in the attempt to present the preliminary propositions rigorously. Whether or not the student can fully comprehend the presentation seems to be a matter of minor importance with some authors. "We must have rigor, absolute exactness, training in logic from the first day on, otherwise," so these dogmatists claim, "the student will be hopelessly led into the habit of slipshod thinking from which no further training can redeem him." This striving for rigor is undoubtedly responsible for the highly artificial character and the complexity of the preliminary propositions as given in a great many textbooks.

In proving, for instance, that the exterior sides (*BA* and *BD*) of supplementary adjacent angles (*ABC* and *CBD*) are in the same straight line, some textbooks produce one of the exterior sides (*AB*) through the vertex, and demonstrate that the prolongation (*BD*′) must coincide with the other exterior side (*BD*) somewhat as follows: The two angles (*ABD* and *ABD*′), being both straight angles, are equal. Subtract from these equal angles the angle *ABC*, and the remainders (*CBD* and *CBD*′) must be equal. Since



angles *CBD* and *CBD'* are equal and have one side (*BC*) common, their other sides (*BD* and *BD'*) must coincide, etc.

Such a proof is intended to introduce a student into the spirit of geometry! Hundreds, if not thousands, of students have been obliged to "know" this proof, *i.e.* to know it by heart, for nobody can comprehend it fully.



If the equality of the large angles (*ABD* and *ABD'*) is admitted at the very start, does not the coincidence of *BD* and *BD'* follow directly from these angles, in precisely the same way as from the last pair of angles (*CBD* and *CBD'*)? What is the object of obtaining the three equations, and subtracting equals from equals? But even this simplification would not make the proof free from objections. Why is one of the exterior sides produced at all? Does not the definition of a straight angle, without any further proof, establish the fact that the exterior sides lie in a straight line?

In a similar spirit many books extend arguments that could be stated in three lines to a length of a page or more, but at the same time commit some blunder that invalidates the entire argument.*

* An instance of this kind, found in a widely used text, is a very long proof of the theorem: From a point without a line, only one perpendicular can be drawn upon the line. The proof rests upon the fact that a certain pair of equal adjacent angles cannot be right angles, and this again is demonstrated by reference to the diagram in which accidentally

A slight knowledge of the foundations of geometry will convince everybody that it is almost impossible to avoid minor inaccuracies in this work, and that considerations that seem to be absolutely flawless fail when scrutinized in the light of modern knowledge. An example of this sort is a proof frequently given for the proposition: One perpendicular can always be erected upon AB at the point A. To prove this assertion, it is *assumed* that AB can be rotated about A until it forms a straight line with its original position. Simple and natural as this assumption appears, it cannot be defended upon purely logical grounds, and a perfectly consistent geometry has been constructed which rejects this possibility.* Of course many will reply: "But we can *see* that such a rotation is possible." This, however, is just the point. "Seeing" is not exact mathematical argument.

Another weakness of these "rigorous" books is the silent assumption of certain theorems, which are by no means more self-evident than those upon which so much time and work are expended. It is inexcusable, the dogmatists tell us, to assume that at a given point only *one* perpendicular can be drawn; but the same critics do not hesitate to assume that every angle has only one bisector — an assumption which includes as a special case the proposition of the perpendicular just mentioned.

the prolongation of one exterior side of one of the angles does not coin cide with the exterior side of the other angle — an assumption which i equivalent to assuming the entire proof.

* Poincaré, Science and Hypothesis, p. 46.

In the hands of a judicious teacher, the harm done by a dogmatic text will be comparatively small, but unfortunately some teachers try to outdo the textbook, and often extend this "absolute exactness" to the mode of reciting. Every minor detail must be given exactly, and the conventional arrangement must be absolutely adhered to. As students are usually unable to do this after one recitation, those demonstrations are repeated again and again until nearly every student gives a "perfect" recitation.

It would of course be absurd to speak against exactness of detail in general, but this is not the place where it should be taught. Exactness in form should be the result of exactness of thinking; and if this latter can be attained, the former will follow as a natural consequence.

EFFECT UPON THE STUDENT*

Wrong impressions at the very start. — To appreciate the difficulties which dogmatic teaching puts into the way of the beginner, we have to realize that the student has to deal here with entirely new ideas, and with methods of reasoning that he never employed before. It is a difficult task for the student, even if the subject is presented pedagogically. But when he is compelled to give an absolutely exact account of logical montrosities, it becomes practically impossible for him to understand, and his failure here, in the beginning, will often affect his whole attitude toward the subject and toward his future work. The student will receive a wrong im-

* Compare Chapter III, The Dogmatic Method.

pression of what geometry really is. He will not see that it is a field for thinking, for invention, for discovery, but he will consider it a system of hair-splitting devices, invented by pedantic schoolmasters for the annoyance of unfortunate students.

Loss of interest. — A normal pupil does not care to study things that he cannot understand, and consequently he will frequently acquire a dislike for the subject at the very start. And it is the more intelligent to whom such work becomes most distasteful. Naturally the interest, which he had during the first hours on account of the novelty of the subject, will soon decrease; and in many cases this interest, this prime stimulus to further good work, will disappear entirely.

Effect upon mode of study. — The most harmful effect of an extreme dogmatic presentation of the preliminary theorems will be the effect upon the student's mode of study. The student, as pointed out above, is unable to see the meaning and the necessity of this so-called rigor. All he can see is that his recitation is never satisfactory unless it is absolutely identical with the statement in the textbook; consequently he studies his demonstrations by heart, and he and his teacher are satisfied. By studying in this way, the student begins to form a habit which will prove to be the most formidable barrier to his further understanding of geometry. He is led to study by using his memory mechanically, instead of relying upon his reasoning power, and if such a habit is once formed, it often makes a pupil unsuccessful in his entire mathematical work.

Importance of initiating students properly into geometry. — It is peculiar that in regard to the study of geometry many students are likely to take extreme positions. Either they like the subject very much, and do good geometric work; or they dislike it extremely, and are then usually unable to do the work properly, although a great deal of memorizing and cramming may disguise this fact. But there is very little middle ground. The start in geometrical study quite often determines these likes and dislikes, and hence every teacher should try his best to initiate the pupils properly. After a student once wakes up to the true meaning, and the beauty and simplicity of mathematics, his further study is more pleasure than work. But there are innumerable students who pass through the entire high school without ever becoming really initiated into mathematical work. Their studies are painful labors, that produce no beneficial and lasting results.

RATIONAL METHODS OF PRESENTING THE PRELIMINARY PROPOSITIONS

Theorems. — If it is impossible to introduce a textbook that presents the matter simply and briefly, the teacher should depart from the text, and simplify matters. If the students are very young and immature, it is advisable not to give proofs for some of these simple theorems, but to assume them as axiomatic.

If we remember that even Euclid assumed that all right angles are equal, there can certainly be no objec-

tion to assuming this or a similar theorem in a secondary school.

Whenever proofs are given, make them as simple and short as possible, and do not wrap the ideas underlying these proofs in a mass of technicalities. "Right angles, being halves of straight angles, are equal, for the halves of equals are equal," and similar statements may not be absolutely accurate, but are sufficient to the beginner, to whom any greater accuracy is meaningless.

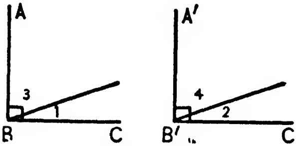
As a rule it is not necessary to insist upon the conventional form in which these matters are represented. Teach the conventional form later, possibly after the propositions on equal triangles, and be satisfied if the students grasp and know the ideas involved in the proofs of the preliminary propositions.

Do not dwell too long upon these preliminary propositions. The longer the teacher dwells upon these and similar simple subjects, the less the class will understand, for many students will after a while suspect difficulties where they do not exist, and will become utterly confused.

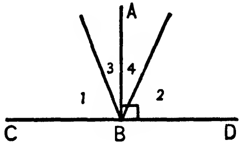
Exercises. — Familiarize the student thoroughly with the meaning of these theorems and give them as many applications as possible. It is somewhat difficult in this connection to form a great many good exercises, but some can be formed. For instance, to familiarize the student with the fact that complements of the same angle or equal angles are equal, the following exercises may be used.

Ex. 1. If ABC and $A'B'C$ are right angles, and $\angle 1 = \angle 2$, prove that $\angle 3 = \angle 4$.

Ex. 2. In the same diagram $AB \perp BC$, $A'B' \perp B'C'$, and $\angle 3 = \angle 4$; prove $\angle 1 = \angle 2$.



Exs. 1 and 2

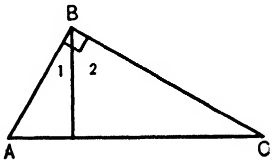


Exs. 3 and 4

Ex. 3. If $AB \perp CD$ and $\angle 1 = \angle 2$, prove $\angle 3 = \angle 4$.

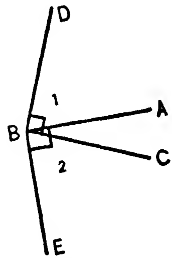
Ex. 4. In the same diagram if $AB \perp CD$ and $\angle 3 = \angle 4$, prove that $\angle 1 = \angle 2$.

Ex. 5. If $\angle ABC$ is a right angle, and $\angle A$ is the complement of $\angle 1$, prove $\angle A = \angle 2$.



Ex. 5

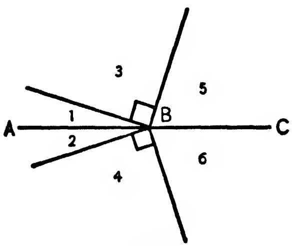
Ex. 6. If DBC and ABE are right angles, prove that $\angle 1 = \angle 2$.



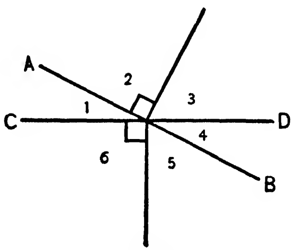
Ex. 6

Ex. 7. If ABC is a st. $\angle$, $\angle 1 = \angle 2$, and $\angle$s 3 and 4 are rt. $\angle$s, prove that $\angle 5 = \angle 6$.

Ex. 8. If AB and CD are straight lines and $\angle$s 2 and 6 are right angles, prove that $\angle 3 = \angle 5$.



Ex. 7



Ex. 8

In a similar way exercises for the supplements of equal angles, etc., can be formed. In particular the exercise that corresponds to Ex. 6, viz. the supplements of the same angle formed by producing its sides, leads to the theorem of vertical angles.

CHAPTER VII

THE ORIGINAL EXERCISE IN GEOMETRY

GENERAL REMARKS

Book proposition or original exercise. — The preceding chapters may be considered as preparatory to the teaching of demonstrative geometry. Before discussing the details of such work it becomes necessary to decide what subject matter must form the bulk of our instruction if we accept the aims of mathematical teaching that are laid down in Chapter II. Should the principal part of the work consist of the study of textbook propositions or the solution of exercises? Should we aim chiefly at the learning of proofs and constructions, or at the ability to do simple original thinking?

Not so very long ago geometric instruction was confined entirely to the former, viz. the learning of textbook demonstrations, while exercises requiring original thought were practically excluded. To-day most schools devote some time to original exercise work, but the manner in which this is sometimes done can hardly be said to be consistent with sound pedagogic principles.

First of all there seems to exist a kind of superstition that the beginner cannot think for himself until he has

mastered a considerable number of formal propositions. In accordance with this view some "standard" textbooks do not give any exercises on the first forty or fifty pages, and some instructors for several months confine themselves to textbook information solely. Some go even farther and recommend first an entire course in plane geometry without exercises, and then a review with "originals." Similarly some textbooks state in their prefaces that all exercises may, or shall, be omitted on the first reading. Thus quite commonly originals are considered as a kind of supplement, a useful but superfluous appendix that has no close connection with the rest of the work.

It seems to the author that such a view of originals is entirely erroneous. To make the exercise an unimportant supplement indicates an absolute lack of understanding of the true function of such work — and we may almost say of the entire object of mathematical instruction.

A course in geometry should be principally a course in the methods of attacking original exercises; the regular book demonstrations should follow as by-products of such a course.

REASONS FOR MAKING THE ORIGINAL EXERCISE THE PRINCIPAL SUBJECT MATTER OF GEOMETRIC INSTRUCTION

1. **Only original thinking represents true geometric work.** — If we concede that it is power and not knowledge that makes the mathematician, and that *thinking*

and not *memorizing* brings into play the beneficial aspects of mathematical study, then the importance of exercise work and the harmfulness of studying a great many ready-made proofs must be admitted.

It has often been remarked that nobody would expect to train a chess player by letting him study the accounts of a great many games, without ever giving him the opportunity to play. The study of models would be of some benefit to the player who has acquired a certain skill in play, provided he had the chance to put such lessons to practical application. The same may be said of any human activity, whether it is skating, baseball playing, or flying. We learn to do by doing.

Only in the study of geometry are students expected to learn to reason, not by practicing reasoning, but by repeating other people's ideas.

2. Exercises form a much better introduction to the study of geometry than does the study of complex models. — It is much easier to solve a simple exercise than to understand a long and complicated proof such as is frequently given on the first pages of textbooks. Exercise work gives the beginner a much better idea of the true character of geometry and prevents him from using mechanical modes of study. Instead of agreeing with the old claim that only the study of several scores of textbook models enables the student to work originals, the writer believes that the opposite is true. The solution of a large number of simple exercises will enable the student to understand clearly and to appreciate model demonstrations.

3. **Exercises arouse the interest** of the student and prevent him from becoming disgusted with the subject. For any normal youth likes to "do," likes to accomplish something. The discovery of a simple mathematical fact is far more interesting, and far more satisfactory, than the studying of pages of information.

4. **Exercises can be much better graded than textbook propositions.** — It is almost impossible to arrange a textbook geometry so that the easiest proofs always occur first, and the rest follow in order of difficulty. Logical sequence must necessarily — more or less, according to the views of the author — disturb the pedagogical sequence. The order of originals, however, may be based upon pedagogic considerations solely.

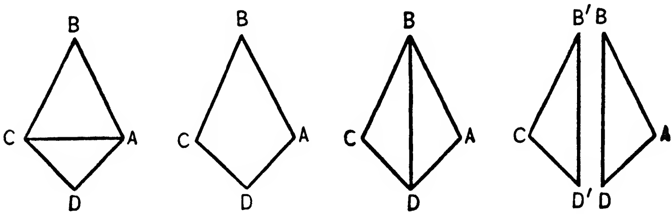
Even the sequence of the regular propositions can be made smoother by inserting exercises that lead from one theorem to another, and thus students may be enabled to discover demonstrations which otherwise would be absolutely beyond their capacity.

How many students would discover the proof for the equality of triangles that are mutually equilateral, if only textbook propositions were studied? It is a simple matter, however, to connect this proposition by a series of exercises with the one that usually precedes it, viz. the base angles of an isosceles triangle are equal. After practicing the demonstration of equal angles by means of this proposition in general, we may give the following questions:

1. If ABC and ADC are two isosceles triangles on the same base, AC, then $\angle BAD = \angle BCD$.

2. If in quadrilateral $ABCD$, $AB = BC$ and $AD = DC$, then $\angle A = \quad C$.

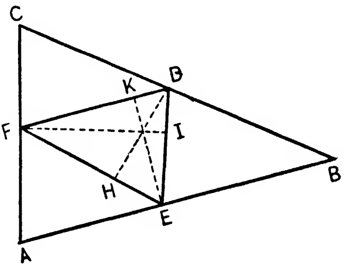
3. If two triangles ABD and DBC have BD common, $AB = BC$ and $AD = DC$, then $\angle A = \angle C$.



4. If in two triangles ABD and $CB'D'$, $AD = CD'$, $AB = A'C$, and $BD = B'D$, then $\angle A = \angle C$.

5. Two triangles ABD and $B'CD'$ are equal if their sides are equal respectively.

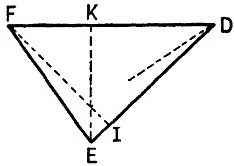
Similarly the concurrence of the three altitudes of a triangle may be found from the concurrence of the three perpendicular bisectors of a triangle as follows: After proving the concurrence of the perpendicular bisectors FI, DH, and EK of the sides of triangle ABC, draw FD, DE, and EF, and ask:



1. What kind of lines are FI, EK, and DH with reference to triangle FDE?

2. What therefore can you tell about the three altitudes of triangle FDE?

3. If any triangle *FDE* were given, could you construct another triangle so that the altitudes of the original triangle would be the perpendicular bisectors of the required triangle?



4. What therefore do you know about the altitudes of any triangle?

Quite often it is an easy matter to discover a proof of a proposition if it is placed immediately after another, while logical considerations make such an immediate succession impossible. The second proposition should then be given as an original immediately after the first, notwithstanding that it occurs again later in the book. Such repetitions are not harmful, but on the contrary very helpful.

Conclusion. — *The most common error of geometric instruction is the fact that the knowledge of book demonstrations is made the chief object of the study.*

The study of geometry should be primarily a course in the solution of originals and general methods of attack. The regular textbook propositions should be treated as exercises, with this difference, that the facts stated by them should be remembered.

Exercises, however, should be studied not in order to be remembered, but in order that the student may familiarize himself with geometric working methods, which will enable him to do other and more complex reasoning.

The student's ability and progress in the subject can be measured only by his ability to solve exercises that are original to him, and not by his ability to repeat well-known facts.

THE TEACHING OF ORIGINAL EXERCISES

Prerequisite on part of the teacher. — If it is granted that the teaching of original exercises is the main object of geometric instruction, then it follows that one of the chief prerequisites for the successful teacher of the subject is the ability to solve exercises easily and rapidly, and this again requires a full knowledge of the various methods of attack.* People whose only mathematical asset is the *knowing* of a certain amount of mathematics, even if this includes a good share of the so-called higher mathematics, are not fit to be teachers of mathematics. A man who is not able to discover such simple matters for himself is certainly not qualified to lead others to such discoveries.

A teacher should also be able to construct geometric exercises in order to meet the particular needs of the moment, or the particular conditions in his classroom. Especially should he be able to extemporize easy oral questions. Such ability is acquired by thinking and by practice, and some concrete directions that are given in later chapters may prove helpful to the beginner.†

Mode of procedure. — While it would be presumptuous to offer advice in this matter to the experienced teacher, a few suggestions may be helpful to the beginner. In many instances it will be advisable to adhere to the following sequence of procedure: (*a*) oral exercises, (*b*) blackboard work, (*c*) individual work on paper. Oral exercises may be largely extem-

* See Chapters XI and XV.

† See pp. 106, 115, 202.

porized by the teacher, who, chalk in hand, may often draw the necessary diagram at the blackboard, and ask questions in rapid succession. Such oral work is well adapted to easy topics, and if skillfully handled, will stimulate interest and arouse competition in the class.

The material for blackboard work, by which is meant here simultaneous work at the blackboard by a considerable number of students may be taken from the textbooks or from sets of previously prepared cards, each of which contains an exercise. Such blackboard work enables the class to cover a considerable amount of ground in a comparatively short time, but it does not produce the same interest and the same rivalry as oral work. It is, however, a much more practical way of correcting the most frequently occurring errors than individual written work. Individual written work will be made easier and its results better by previous blackboard work.

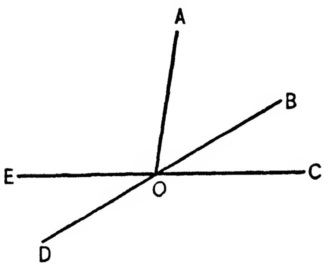
Construction of Exercises. — It would be useless to enlarge any further upon the general directions for the teaching of exercises. Concrete illustrations are far more serviceable, and the next chapters are almost exclusively devoted to these. Here we shall discuss and give exercises illustrating only the first regular proposition,* viz. *Vertical angles are equal.*

It was shown on page 97 in what manner the preliminary propositions may be used to discover the proof of the equality of vertical angles. In case this should be too difficult for some students, it may be made still

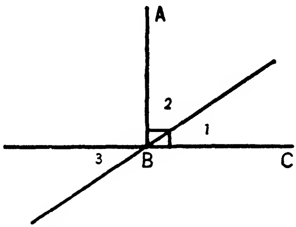
* The sequence of propositions assumed here is that of Schultze and Sevenoak's Geometry.

easier by means of numerical examples. Assign a numerical value to one angle of the diagram and let the student find the values of the other, making, of course, no use of the equality of vertical angles. After a few examples of this kind, assign a literal value, *e.g.* $n°$, to one angle and proceed as before, thereby establishing the equality of a pair of vertical angles.

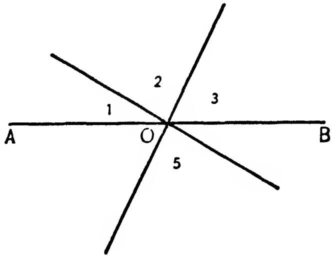
To construct exercises which illustrate the equality of vertical angles, take almost any exercise of the preceding pages, and vary it by introducing in place of one or of several angles their vertical angles.



Thus in the annexed diagram instead of asking, "Which angle is the sum of $\angle$s AOB and BOC?" introduce $\angle EOD$, the vertical angle to BOC, and ask, "Which angle is the sum of $\angle$s AOB and DOE?" Similarly in the same diagram ask for the difference of AOC and DOE. In the next diagram let $AB \perp BC$. Instead of asking for the complement of $\angle$ 1, ask for the complement of $\angle$ 3. In the next diagram (three straight lines meeting at O), instead of asking for the sum of $\angle$s 1, 2, and 3, introduce $\angle$ 5, the vertical angle to 2, and ask for the sum of angles 1, 5, and 3.



Each of these exercises leads to numerical questions. So in the last we may assign numerical values to angles 1 and 3, and may require the value of 5.



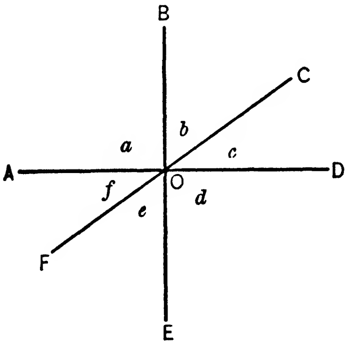
It is a very simple matter to construct exercises of this kind, and the following paragraph contains a considerable number of originals obtained in this manner.

EXERCISES

Let three straight lines AD, BE, and CF meet in O, forming the six angles, a, b, c, d, e, and f.

Ex. 1. If $\angle b = 40°$, and $\angle c = 35°$, find $\angle AOE$.

Ex. 2. If $\angle FOB = 130°$, and $\angle c = 40°$, find $\angle a$.



Ex. 3. If $\angle FOB = 130°$, and $\angle f = 38°$, find $\angle d$.

Ex. 4. If $\angle f = 60°$, and $\angle b = 25°$, find $\angle d$.

Ex. 5. If ∠s b and f are complementary, find $\angle d$.

Ex. 6. If $\angle AOC = 140°$, and $\angle COE = 120°$, find $\angle BOD$.

Ex. 7. If $\angle AOC = 150°$, and $\angle COE = 130°$, find $\angle a$.

Ex. 8. If $\angle FOB = 140°$, and $\angle AOC = 125°$, find $\angle d$.

Ex. 9. If $\angle AOE + \angle BOC = 140°$, and $\angle c = 40°$, find $\angle e$.

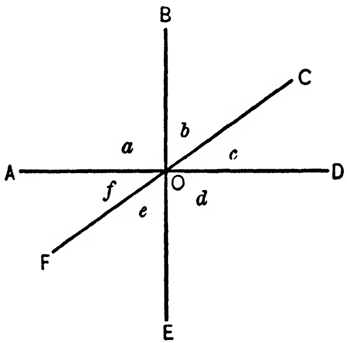
Ex. 10. If $\angle f = \angle b$, and $\angle d = 100°$, find $\angle c$.

Ex. 11. If $\angle b = \angle c$, and $\angle AOE = 80°$, find $\angle f$.

Ex. 12. If $\angle AOC = \angle BOD$, and $\angle e = 40°$, find $\angle a$.

Ex. 13. If $\angle a = 2(\angle c)$, and $\angle e = 60°$, find $\angle c$.
Ex. 14. If $\angle AOC = \angle COE$, and $\angle BOD = 100°$, find $\angle f$.

If we bear in mind that in all the problems relating to the preceding diagram we have two independent unknown quantities, it is obvious that two equations connecting independent angles will determine all other angles in the above diagram. Thus we may increase the number of exercises almost without limit, *e.g.*



Ex. 15. If $\angle a - \angle b = 15°$, and $\angle c = \angle d$, find $\angle c$.

Ex. 16. If $\angle a = 2(\angle b)$, and $\angle c + \angle e - \angle d = 20°$, find $\angle b$.

Ex. 17. If $\angle a - \angle b = 10°$, and $\angle d - \angle c = 20°$, find $\angle c$.

The number of exercises may be still further increased by noting that in each of the preceding examples not only the one required angle, but every angle of the diagram, may be found. We may also vary the above examples by expressing the given quantities not in numbers, but in letters, as $m°$ or $n°$.

Finally, each of these numerical problems may be changed into a theorem, thus producing another set of more difficult exercises. *E.g.* Exs. 1, 2, 3, 4, 6, and 7 lead to the following theorems:

Ex. 18. Prove that $\angle b + \angle c = \angle AOE$.
Ex. 19. Prove that $\angle FOB - \angle c = \angle a$.
Ex. 20. Prove that $\angle FOB - \angle f = \angle d$.
Ex. 21. Prove that $\angle f + \angle b + \angle d = 180°$.

Ex. 22. Prove that $\angle AOC + \angle BOD + \angle COE = 360°$.

Ex. 23. Prove that $\angle COE - \angle a = 180° - \angle AOC$, etc.

A few other theorems may follow:

Ex. 24. If $\angle f = \angle e$, prove that $\angle b = \angle c$.

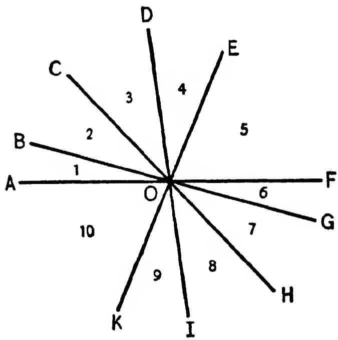
Ex. 25. If $\angle FOB = \angle FOD$, prove that $\angle f = \angle e$.

Ex. 26. If $\angle a - \angle b = \angle d - \angle c$, prove that $\angle b = \angle e$.

Ex. 27. Prove that $\angle AOC + \angle COE - \angle EOA = 2(\angle a)$.

Ex. 28. Prove that reflex $\angle AOE$ + reflex $\angle BOF$ + reflex $\angle COA = 720°$.

Similar exercises may be formed if four,* five, or more lines meet in a point, but the preceding illustrations seem to make it superfluous to give many additional examples. Hence only a few will be given, all of them relating to five lines meeting at a point O, and forming the ten angles 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10.



Ex. 29. Prove that $\angle 1 + \angle 3 + \angle 5 + \angle 7 + \angle 9 = 180°$.

Ex. 30. Prove that $\angle AOC + \angle BOD + \angle COE + \angle DOF + \angle EOG = 360°$.

Ex. 31. Prove that $\angle AOD + \angle BOE + \angle COF + \angle DOG + \angle EOH = 540°$.

Ex. 32. Form a similar proposition for $\angle$s AOE, BOF, etc.

Ex. 33. If $\angle AOC = \angle BOD$, prove that $\angle 6 = \angle 8$.

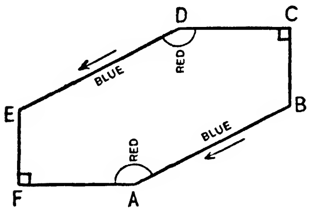
Ex. 34. If $\angle AOC = \angle BOD = \angle COE = \angle DOF = \angle EOG$, then all ten angles (1, 2 ···) are equal.

Ex. 35. If $\angle AOD = \angle BOE = \angle COF = \angle DOG = \angle EOH$, then $\angle 1 = \angle 2 = \angle 3 = \angle 4$, etc.

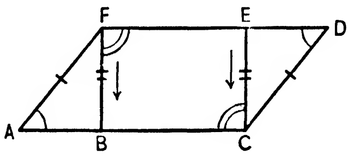
* The diagram formed by 4 straight lines contains 3 independent unknown quantities, hence offers a wide field for practicing the solution of simultaneous equations involving 3 unknown quantities.

GRAPHIC METHODS FOR PRESENTING GEOMETRIC FACTS

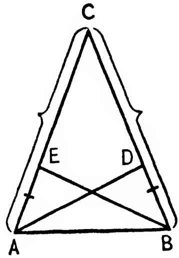
To teach students to think, we should in the early stages remove as far as possible all external difficulties. Students who can reason logically sometimes forget the hypothesis, or forget preceding parts of the proof, and hence are unable to continue. To make such forgetfulness almost impossible graphic methods may be employed. The hypothesis may be indicated by colors, equal colors representing equal lines or equal angles, arrows denoting parallel lines, a small colored square indicating a right angle. Thus, in the annexed diagram the two blue lines indicate the equality of AB and DE, the two red arcs indicate the equality of ∠s A and D, the two colored arrows represent the parallelism of AB and DE, and the two small squares show that ∠s F and C are right angles.



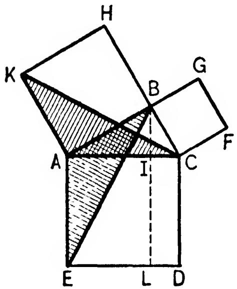
For the results obtained in the proof we use white crayon, equality of lines being indicated by equal cross-marks, equality of angles by equal number of arcs, parallelism by arrows, etc. Thus the marks in the annexed diagram, which are supposed to be drawn in white crayon, indicate that we have proved: $AF = CD$, $BF = CE$, $\angle A = \angle D$,



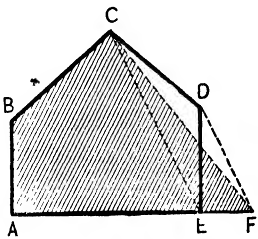
$\angle BFE = \angle ECB$, $BF \parallel CE$. If lines overlap, we use braces. Thus the marks of the annexed diagram represent the equality of AC and BC, and the equality of AE and BD.

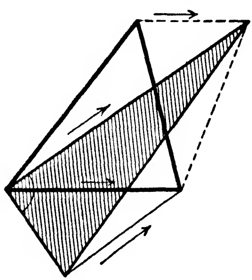


If colored crayons are not available, white cross marks, etc., may also be used to indicate the hypothesis, but for the beginner a distinction between hypothesis and proof is very helpful.



To point out certain triangles or polygons to the students, either shade them or mark their perimeters by heavy lines, as $\triangle$ AEB and KAC in the annexed diagram. Thus, if we transform one figure into another, it is advisable to mark the perimeter of the given area heavy, and to shade the resulting area, as indicated in the annexed diagrams.





In other constructions given lines may be drawn thin, lines of construction dotted, and resulting lines heavy.

In complex constructions various colors may be used to distinguish among different lines.

It is impossible to mention all such cases, and the resourceful teacher will have no difficulty in enlarging and modifying the above directions.

CHAPTER VIII

EQUALITY OF TRIANGLES

THE FIRST TWO PROPOSITIONS OF EQUAL TRIANGLES

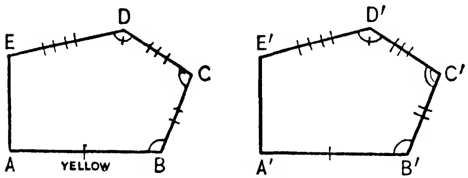
Can superposition be avoided? — Proofs by superposition as a rule do not appeal to students, and as they occur only rarely, the question is sometimes asked whether it is possible to devise a system of demonstrations without the use of superposition. If we consider, however, that the first demonstration of equality which refers to any particular figure can be based upon nothing but the definition of equality, and that this definition involves superposition, it becomes evident that superposition cannot be avoided unless we change the definition of equality.* Hence the first proposition of equal triangles, or equal arcs, or equal ellipses, etc., must necessarily be based upon superposition.

The study of superposition. — The method of making the student familiar with the proofs for the equality of triangles by means of frequent repetition, can hardly be recommended. Rather impart to him by concrete

* This remark refers to the usual school geometry only. There are systems of geometry that dispense entirely with the axiom of mobility and hence superposition. Such systems, however, require additional axioms, and the first theorem of equal triangles is usually made one of these axioms.

illustrations a knowledge of the fundamental mode of procedure, viz. the superposition of some parts whose equality is known, and the successive tracing of the remaining parts of the figure. This may be accomplished by exercises like the following:

In pentagons $ABCDE$ and $A'B'C'D'E'$, let $AB = A'B'$, $BC = B'C'$, $CD = C'D'$, $DE = D'E'$, $\angle B = \angle B'$, $\angle C = \angle C'$, and $\angle D = \angle D'$. Draw one figure, as $ABCDE$, in some color, *e.g.* in yellow, and indicate the given equalities by cross marks. When the student applies AB to $A'B'$, let him draw a yellow



line on top of $A'B'$ to indicate real physical superposition, and so forth for all other parts. After a few exercises of this type, the student will understand superposition, and will be able to apply it to the two triangle propositions (*a.s.a.* = *a.s.a.* and *s.a.s.* = *s.a.s.*).

It is, however, not advisable to dwell too long upon these proofs and to make them distasteful to the student by frequent repetition and pedantic insistence upon unimportant detail. Rather pass over this topic hastily, even if not every student can give a perfect recitation.

THE TEACHING OF ORIGINALS BASED UPON EQUAL TRIANGLES

General remarks. — After the class understands the meaning of the first two propositions on equal triangles,

easy oral exercises should be attacked. The teacher may draw the diagram on the blackboard, indicating the hypothesis by colored crayons.* Then the fact should be brought out that equal triangles may be used to demonstrate the equality of lines and angles, and by a great many originals this should be made familiar to the student. To cover a large number of such exercises simultaneous blackboard work is necessary. The main result of all exercises should be the knowledge of

METHOD I. **The equality of lines and angles is usually proved by means of equal triangles.**

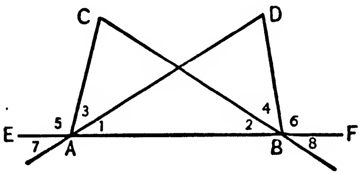
The teacher cannot emphasize this principle and its application too much. Its knowledge is far more important than the knowledge of any regular proposition. Whenever the student is required to prove the equality of lines or angles, the analysis should be started by the question — first asked by the teacher, later by the student: "What is the usual method of demonstrating the equality of lines and angles?" There are only a few such principles that deserve the same emphasis, *e.g.* the methods for demonstrating parallelism of lines, inequality of lines, proportionality of lines, equivalence of areas, and a few others. A full knowledge of these principles and their applications forms one of the chief aims of geometric study. It would, however, be a mistake to base a "method" upon almost every theorem in the textbook.

Method of constructing exercises. — It is a comparatively simple matter to construct a great many originals

* See Chapter VII, page 110.

illustrating the application of equal triangles. Select two triangles in certain relative positions, as ABC and ABD, and determine by which methods we may obtain the equality of two sides and the included angle, or the equality of two angles and the included side.

Angles may be equal by hypothesis, as complements of equals, as supplements of equals, as vertical angles, as sums of equals, etc. Lines may be equal by hypothesis or by the axioms of the sums or differences of equals, etc.



To obtain an application of the theorem ($a.s.a. = a.s.a.$) in the preceding diagram, we consider that AB is common, hence we have to find various ways of making $\angle 1 = \angle 2$, and $\angle CAB = \angle DBA$. Introducing $\angle$s 7 and 8, the vertical $\angle$s of 1 and 2, we get the following hypotheses.

1. $\angle 1 = \angle 2$, $\angle CAB = \angle DBA$.
2. $\angle 1 = \angle 2$, $\angle 3 = \angle 4$.
3. $CA \perp AB$, $DB \perp AB$, $\angle 1 = \angle 2$.
4. $CA \perp AB$, $DB \perp AB$, $\angle 3 = \angle 4$.
5. $CA \perp AB$, $DB \perp AB$, $\angle 1 = \angle 3$, $\angle 2 = \angle 4$.
6. $\angle 1 = \angle 2$, $\angle 5 = \angle 6$.
7. $\angle EAD = \angle CBF$, $\angle 3 = \angle 4$.
8. $\angle 3 = \angle 4$, $\angle 5 = \angle 6$.
9. $\angle 1 = \angle 8$, $\angle CAB = \angle ABD$.
10. $\angle 1 = \angle 8$, $\angle 3 = \angle 4$.
11. $\angle 7 = \angle 2$, $\angle 5 = 6$.
12. $\angle 7 = \angle 8$, $\angle CAD = \angle CBD$.
13. $AC \perp AB$, $DB \perp AB$, $\angle 7 = \angle 2$.

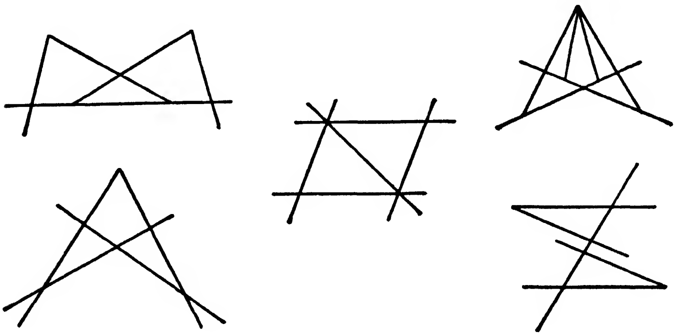
Similarly we may illustrate ($s.a.s. = s.a.s.$) by the following exercises:

14. $AC = DB$, $\angle CAB = \angle DBA$.
15. $AC = DB$, $\angle 5 = \angle 6$.
16. $AD = BC$, $\angle 1 = \angle 8$.
17. $AD = BC$, $\angle 7 = \angle 8$.
18. $AC = DB$, $AC \perp AB$, $DB \perp BA$.

This list could be considerably extended by introducing the vertical angles of 3 and 4, or of CAB and DBA. Still more such exercises may be obtained by varying the conclusion, thus requiring in one exercise the equality of $\angle$s C and D, in another the equality of AC and BD, etc.

Hence one diagram will furnish us with a great many theorems, and as it is an easy matter to find other such diagrams, it must be admitted that there is no lack of material.

A few such diagrams are given below; others may be found in the next paragraph.



List of exercises.—As hardly any textbook contains a sufficient number of exercises of this type, and as the subject is of utmost importance, the list given below has been made very extensive.

It is of course not necessary for each individual to solve all the exercises, but for simultaneous blackboard work a large number of questions are needed. As stated in the preceding chapter, it is sometimes advisable to prepare cards, each containing a problem, for rapid distribution of the questions.*

Examples on equal triangles may be divided into six classes:

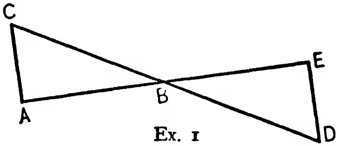
1. Numerical examples.
2. Triangles that do not overlap.
3. Triangles that overlap.
4. Student has to choose between several pairs of equal triangles.
5. The triangles must be constructed.
6. Several pairs of equal triangles are necessary for the proof.†

1. *Numerical exercises.* — If the pupils of a class are very immature, this subject may be commenced with numerical exercises. Only a few are given here; but if more should be needed, they can be obtained by substituting equal numerical values for any equal lines or angles in any of the exercises following.‡

* The exercises are grouped according to diagrams, in order to avoid an unnecessary multiplication of figures. In the classroom, exercises referring to one figure should not be studied in immediate succession.

† Only five of these six classes are given in the following list; for the last one, see p. 135.

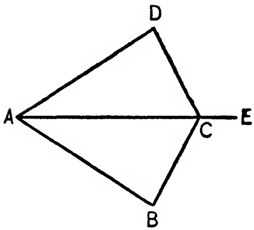
‡ In the following list, if the nature of the diagram shows that certain lines are intended to be straight ones, this fact has — for the sake of brevity — not always been stated. The teacher, in using such exercises, however, had better add this to the hypothesis.



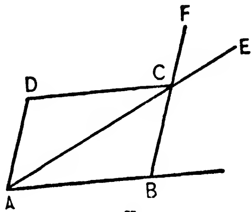
Ex. 1

Ex. 1. If $AB = 8$ inches, $BE = 8$ inches, $\angle A = 80°$, $\angle E = 80°$, prove $\triangle ABC = \triangle BDE$.

Ex. 2. If $\angle DAC = 30°$, $\angle CAB = 30°$, $\angle DCE = 130°$, $\angle ACB = 50°$, prove $\triangle ABC = \triangle ACD$.



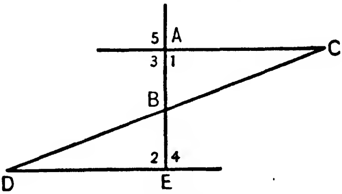
Ex. 2



Ex. 3

Ex. 3. If $\angle DAC = 40°$, $\angle FCE = 40°$, $\angle DCA = 30°$, and $\angle CAB = 30°$, prove $\triangle ABC = \triangle ACD$.

2. *Triangles that do not overlap.* —



Exs. 4–8

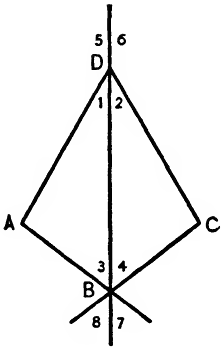
Ex. 4. If $AB = BE$, and $\angle 1 = \angle 2$, prove $\triangle ABC = \triangle BDE$.

Ex. 5. If $AB = BE$, $AC \perp AE$, and $DE \perp AE$, prove $\triangle ABC = \triangle BDE$.

Ex. 6. If $AB = BE$, and $CB = BD$, prove $\triangle ABC = \triangle BDE$.

Ex. 7. If $AB = BE$, and $\angle 3 = \angle 4$, prove $BC = BD$.

Ex. 8. If $AB = BE$, and $\angle 5 = \angle 2$, prove $\angle C = \angle D$.



Exs. 9–13

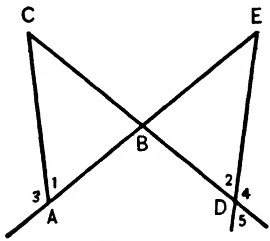
Ex. 9. If $AD = DC$, and $\angle 1 = \angle 2$, prove $AB = BC$.

Ex. 10. If $AD = DC$, and $\angle 5 = \angle 6$, prove $\angle A = \angle C$.

Ex. 11. If $\angle 3 = \angle 4$, and $\angle 5 = \angle 6$, prove $AD = DC$.

Ex. 12. If $AB = BC$, and $\angle 4 = \angle 6$, prove $AD = DC$.

Ex. 13. If $\angle 5 = \angle 6$, and $\angle 7 = \angle 8$, prove $\angle A = \angle C$.



Exs. 14–18

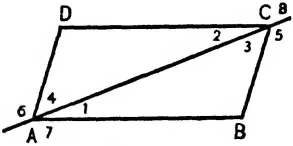
Ex. 14. If $AB = BD$, and $BC = BE$, prove $CA = ED$.

Ex. 15. If $AB = BD$, $CA \perp AB$, and $ED \perp BD$, prove $\angle C = \angle E$.

Ex. 16. If $AB = BD$, and $\angle 3 = \angle 4$, prove $CA = ED$.

Ex. 17. If $AE = DC$, and $BE = CB$, prove $\angle 1 = \angle 2$.

Ex. 18. If $AB = BD$, and $\angle 1 = \angle 5$, prove $BC = BE$.



Exs. 19–23

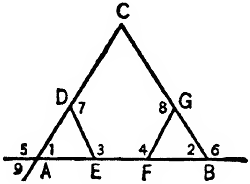
Ex. 19. If $\angle 1 = \angle 2$, and $\angle 5 = \angle 6$, prove $AB = CD$.

Ex. 20. If $\angle DAB = \angle BCD$, and $\angle 1 = \angle 2$, prove $AD = BC$.

Ex. 21. If $AD = BC$, and $\angle 6 = \angle 5$, prove $\angle B = \angle D$.

Ex. 22. If $\angle 5 = \angle 6$, $\angle 7 = \angle 8$, prove that $AD = BC$.

Ex. 23. If $\angle 6 = \angle 5$, $\angle DAB = \angle DCB$, prove that $AB = DC$.



Exs. 24–28

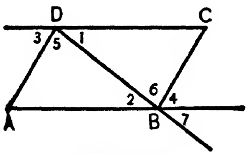
Ex. 24. If AB is trisected, $\angle 1 = \angle 2$, $DE \perp AB$, and $GF \perp AB$, prove that $DE = GF$.

Ex. 25. If $AE = FB$, $\angle 1 = \angle 2$, $\angle 3 = \angle 4$, prove that $AD = BG$.

Ex. 26. If $\angle 5 = \angle 6$, $\angle 7 = \angle 8$, and $AD = GB$, prove that $AE = BF$.

Ex. 27. If $AD = BG$, $AF = EB$, and $\angle 1 = \angle 2$, prove that $DE = GF$.

Ex. 28. If $AF = EB$, $AD = BG$, and $\angle 2 = \angle 9$, prove that $ED = FG$.



Exs. 29–33

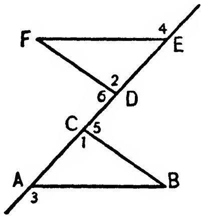
Ex. 29. If $AD \perp DB$, $BC \perp BD$, and $\angle 1 = \angle 2$, prove that $\angle A = \angle C$.

Ex. 30. If $AD \perp DB$, $BC \perp BD$, and $DA = BC$, prove that $AB = DC$.

Ex. 31. If $\angle 1 = \angle 2$, and $\angle 3 = \angle 4$, prove that $DA = CB$.

Ex. 32. If $\angle 3 = \angle 4$, and $\angle 5 = \angle 6$, prove that $AB = DC$.

Ex. 33. If $\angle 1 = \angle 7$, and $\angle 5 = \angle 6$, prove that $DA = BC$.



Exs. 34–38

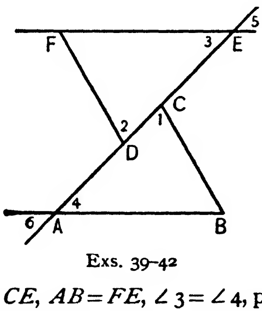
Ex. 34. If $AD = CE$, $FD = BC$, and $\angle 1 = \angle 2$, prove that $\angle B = \angle F$.

Ex. 35. If $AD = CE$, $\angle 1 = \angle 2$, $\angle 3 = \angle 4$, prove that $AB = FE$.

Ex. 36. If $AD = CE$, $FD = BC$, and $\angle 5 = \angle 6$, prove that $\angle B = \angle F$.

Ex. 37. If $AD = CE$, $FD \perp AE$, $BC \perp AE$, and $\angle 3 = \angle 4$, prove that $FD = BC$.

Ex. 38. If $AD = CE$, $AB = FE$, and $\angle 3 = \angle 4$, prove $\angle B = \angle F$.



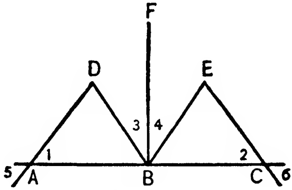
Exs. 39–42

Ex. 39. If $AD = CE$, $AB = FE$, $\angle 3 = \angle 4$, prove that $\angle B = \angle F$.

Ex. 40. If $AD = CE$, $AB = FE$, $\angle 5 = \angle 6$, prove that $BC = FD$.

Ex. 41. If $AD = CE$, $FD \perp AE$, $BC \perp AE$, and $\angle 4 = \angle 5$, prove that $AB = FE$.

Ex. 42. If $AD = CE$, $FD \perp AE$, $BC \perp AE$, and $FD = BC$, prove that $\angle B = \angle F$.



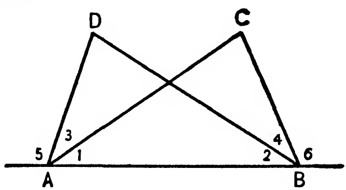
Exs. 43–45

Ex. 43. If $FB \perp AC$, $\angle 3 = \angle 4$, B is the mid-point of AC, and $\angle 1 = \angle 2$, prove that $\angle D = \angle E$.

Ex. 44. If $FB \perp AC$, $\angle 3 = \angle 4$, $AB = BC$, and $\angle 5 = \angle 6$, prove that $AD = CE$.

Ex. 45. If $AD \perp DB$, $FB \perp AC$, $CE \perp EB$, $\angle 3 = \angle 4$, and $BD = BE$, prove $AB = BC$.

3. *Triangles that overlap.*—The following examples contain triangles that overlap, and each diagram contains several equal triangles. The student has to choose *that pair of triangles which contains the angles or sides to be proved equal.*



Exs. 46–51

Ex. 46. If $AD \perp AB$, $CB \perp AB$, and $AD = BC$, then $\triangle ABD = \triangle ABC$.

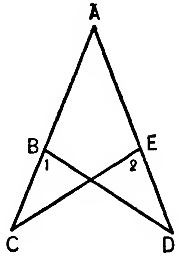
Ex. 47. If $AD \perp AB$, $CB \perp AB$, and $\angle 3 = \angle 4$, then $AC = DB$.

Ex. 48. If $\angle 5 = \angle 6$, and $AD = CB$, then $\angle 1 = \angle 2$.

Ex. 49. If $\angle 5 = \angle 6$, and $\angle 1 = \angle 2$, then $AC = DB$.

Ex. 50. If $\angle 5 = \angle 6$, and $\angle 3 = \angle 4$, then $AC = DB$.

Ex. 51. If $\angle 5 = \angle 6$, AC bisects $\angle DAB$, and BD bisects $\angle ABC$, then $AC = BD$.



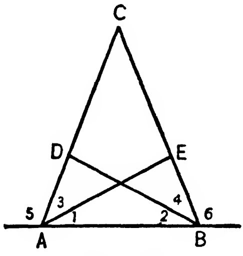
Exs. 52–55

Ex. 52. If $AB = AE$, and $BC = ED$, then $\triangle ABD = \triangle ACE$.

Ex. 53. If $AB = AE$, and $\angle 1 = \angle 2$, then $CE = BD$.

Ex. 54. If $AC = AD$, and $BC = ED$, then $CE = BD$.

Ex. 55. If B is the mid-point of AC, E the mid-point of AD, and $AB = AE$, then $\angle ABD = \angle AEC$.



Exs. 56–61

Ex. 56. If $\angle 1 = \angle 2$, and $\angle 3 = \angle 4$, then $\triangle ADB = \triangle AEB$.

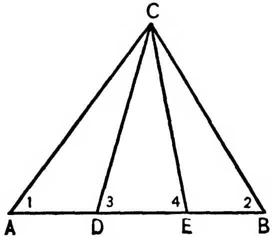
Ex. 57. If $\angle DAB = \angle EBA$, and $\angle 3 = \angle 4$, then $\triangle ADB = \triangle AEB$.

Ex. 58. If $\angle 5 = \angle 6$, and $\angle 1 = \angle 2$, then $\triangle ADB = \triangle AEB$.

Ex. 59. If $\angle 5 = \angle 6$, and $AD = BE$, then $\angle 1 = \angle 2$.

Ex. 60. If $\angle 5 = \angle 6$, AE bisects $\angle DAB$, and BD bisects $\angle ABE$, then $AE = BD$.

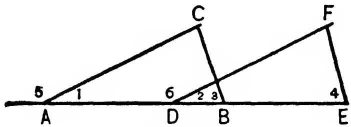
Ex. 61. If $AC = BC$, D is the mid-point of AC, and E is the mid-point of BC, then $\angle 3 = \angle 4$.



Exs. 62, 63

Ex. 62. If $AD = EB$, $\angle 1 = \angle 2$, and $\angle 3 = \angle 4$, then $\triangle AEC = \triangle DBC$.

Ex. 63. If $AD = EB$, $CD = CE$, and $\angle 3 = \angle 4$, then $\triangle AEC = \triangle DBC$.

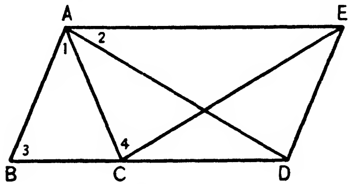


Exs. 64–66

Ex. 64. If $AD = BE$, $\angle 1 = \angle 2$, and $\angle 3 = \angle 4$, then $BC = FE$.

Ex. 65. If $AD = BE$, $\angle 5 = \angle 6$, and $\angle 3 = \angle 4$, then $AC = DF$.

Ex. 66. If $AD = BE$, $AC = DF$, and $\angle 5 = \angle 6$, then $\angle C = \angle F$.

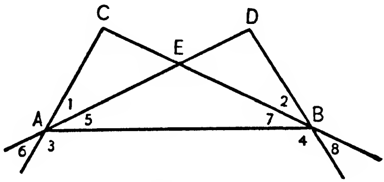


Exs. 67, 68

Ex. 67. If BCD is a straight line, $AB = AC$, $AD = AE$, and $\angle 1 = \angle 2$, then $BD = CE$.

Ex. 68. If $\angle 1 = \angle 2$, $\angle 3 = \angle 4$, $AB = AC$, and BCD is a straight line, then $AD = AE$.

4. *The proper triangles must be found by trial.* — In the following exercises the two parts whose equality is to be demonstrated may be considered homologous parts of two different pairs of triangles, and the student has to determine by trial which pair must be used.



Exs. 69–73

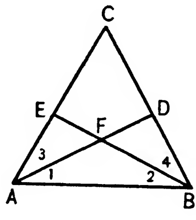
Ex. 69. If $AE = BE$, and $\angle 1 = \angle 2$, then $\angle C = \angle D$.

Ex. 70. If $\angle 5 = \angle 7$, and $\angle 1 = \angle 2$, then $\angle C = \angle D$.

Ex. 71. If $\angle 1 = \angle 2$, and $\angle 3 = \angle 4$, then $\angle C = \angle D$.

Ex. 72. If $DA = CB$, and $CE = DE$, then $\angle C = \angle D$.

Ex. 73. If $\angle 6 = \angle 5$, $\angle 7 = \angle 8$, $\angle 3 = \angle 4$, then $\angle C = \angle D$.



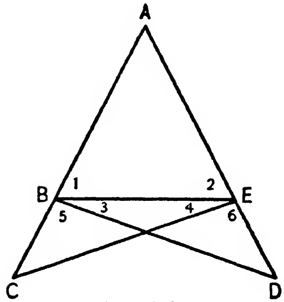
Exs. 74–77

Ex. 74. If $\angle EAB = \angle DBA$, and $\angle 3 = \angle 4$, then $AD = BE$.

Ex. 75. If $AC = BC$, and $\angle 3 = \angle 4$, then $AD = BE$.

Ex. 76. If $\angle EAB = \angle DBA$, $\angle 1 = \angle 3$, and $\angle 2 = \angle 4$, then $AD = BE$.

Ex. 77. If $CA = CB$, and AD and BE are medians, then $AD = BE$.



Exs. 78–84

Ex. 78. If $\angle 1 = \angle 2$, $DB \perp AB$, and $CE \perp AE$, then $\angle C = \angle D$.

Ex. 79. If $\angle 1 = \angle 2$, and $\angle 3 = \angle 4$, then $EC = BD$.

Ex. 80. If $AB = AE$, and $\angle 5 = \angle 6$, then $EC = BD$.

Ex. 81. If $\angle 1 = \angle 2$, and $\angle 5 = \angle 6$, then $\angle C = \angle D$.

Ex. 82. If $AB = AE$, and $BC = ED$, then $\angle C = \angle D$.

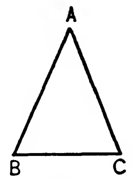
Ex. 83. If $\angle 1 = \angle 2$, $\angle 3 = \angle 5$, $\angle 4 = \angle 6$, then $BD = EC$.

Ex. 84. If $\angle C = \angle D$, $CB = ED$, and $AB = AE$, then $CE = BD$.

5. *The triangles must be constructed.* — This mode of procedure may be stated as:

METHOD II. **If the lines or angles whose equality we wish to demonstrate are not parts of equal triangles, we have to make them parts of equal triangles by drawing additional lines.**

This is an important principle that is used extensively in succeeding chapters. At this point it is difficult to form exercises, but after studying parallelism of lines and the remaining propositions on equal triangles, it is a very simple matter to form such examples. Of the four exercises following, only one can be done by the student at this stage of the work.



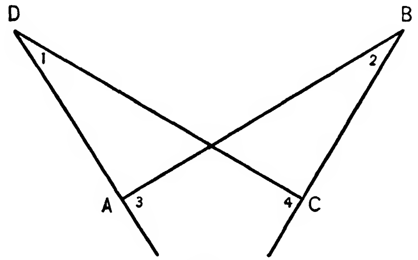
Ex. 85

Ex. 85. If $AB = AC$, then $\angle B = \angle C$.

Ex. 86. If in quadrilateral $ABCD$, $AB = CD$, and $BC = DA$, then $\angle A = \angle C$.

Ex. 87. If in quadrilateral $ABCD$, $\angle s\ A$ and C are rt. $\angle s$, and $AB = DC$, then $BC = DA$.

Ex. 88. If $AB = CD$, and $\angle 1 = \angle 2$, then $\angle 3 = \angle 4$.



Ex. 88

All the preceding exercises give both hypothesis and conclusion explicitly. By degrees, however, the student must become trained to understand exercises stated in more complex form. *E.g.*—

Ex. 89. If a diagonal of a quadrilateral bisects the angles whose vertices it joins, the figure is divided into two equal triangles.

Ex. 90. If, from the ends of the base of an isosceles triangle, equal distances be laid off on the arms, and from their ends lines be drawn to the opposite vertices, these lines are equal.

Importance of the exercises on equal triangles. — There is hardly another topic in the study of geometry that deserves more attention, and that is more important to the student than the applications of the equality of triangles. For, first, this method is used more than any other in the succeeding chapters of plane geometry. Secondly, there is no other topic that introduces the student so fully and easily into the true spirit of geometric work.

It is a fact that many students finish courses in geometry and pass examinations without ever becoming fully initiated into the real meaning of geometric work. The present chapter forms a critical point. Here the student should become fully aware what geometry really means and what sort of mental activity is necessary for its study.

Hence the teacher should dwell upon this topic, and not proceed until every student who possesses normal reasoning powers has fully mastered it.

CONVENTIONAL METHOD OF STATING GEOMETRIC PROOFS

Since the majority of the preceding examples cannot be done orally, but must be written, it becomes necessary to study the various conventional ways of stating geometric proofs.

Symbols or words? — Although the extensive use of symbols in geometry is at the present day almost general, still there remain a few teachers and a few books that employ hardly any symbols, claiming that

the teaching of English is one of the chief objects of geometric instruction.

It is, however, entirely unpedagogical to accumulate difficulties. The use of symbols makes the study of geometry easier; hence let us employ them as far as they have been generally accepted. Our first aim must be the teaching of geometry, and the effect this teaching has upon the student's linguistic abilities is — while important — only of secondary importance. Success in the first aim is the absolute prerequisite of success in the second.*

Symbols of uncertain and varying significance. — A remarkable feature of mathematical symbolism is the fact that it forms an international language that can be understood by any mathematician, no matter what his nationality. Unfortunately there is no absolute agreement in regard to a few symbols, the most widely known examples being the symbols for equality, similarity, and equivalence. Here not only symbols but even terms differ. In America the equality of areas is called equivalence and rēpresented by the symbol $\backsimeq$, while in continental Europe it is called equality and denoted by $=$. Figures that may be superposed are called in

* Moreover, geometric study does far more for the mastery of the mother tongue — using symbols, or no symbols — than generally supposed. It does so by virtue of its accuracy and precision, and by improving in general the student's ability to think. For it is an undeniable fact that many pupils cannot speak correctly because they cannot think correctly. They constantly use words and phrases without having any definite ideas to express. (Compare Chapter II.)

America equal ($=$), in the greater part of Europe congruent ($\cong$). The symbol for similarity ($\sim$)* generally employed in Europe is used very little in America.

The chief trouble arises from the use of the term "equality" in geometry. The American use is provincial and illogical, for it gives a meaning to this word in geometry entirely different from that in the rest of the mathematical subjects. Equality, thus used, refers in geometry to form and size, while in arithmetic or algebra it has no bearing upon form. "If equals be added to equals the sums are equal" would be true in an algebraic sense only, but erroneous when applied to geometric figures. Besides the symbol $\Bumpeq$ is a difficult character to write. A change to the symbols $=$, $\sim$, and $\cong$, and the corresponding terms, is certainly most desirable.

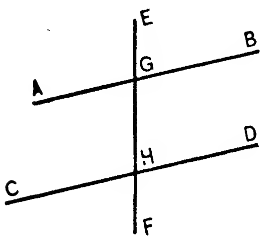
Another provincial symbol that should be discarded is $::$, as used in proportion. The true meaning of this symbol is equality, and hence it should be replaced by $=$. The mere fact that it is read "as" does not justify the introduction of a new symbol. If we should use different symbols whenever the words differ which these symbols represent, we would have to use at least half a dozen symbols in place of "minus" ($-$), just as many for "equals" ($=$) etc.

Certain other symbols have different meanings in different parts of mathematics, *e.g.* the symbol $\equiv$. In the theory of numbers it denotes congruence of numbers, in algebra it is used to represent an identity.

* Leibnitz introduced the symbol $\sim$, and this is the most widely used form, but some authors use $\backsim$ instead in order to connect the symbol with the first letter of the word (similis).

The statement of the hypothesis. — There are two different ways of stating the hypothesis. The American method describes the given part so that a reader can understand it without seeing a diagram. He could draw the diagram from the hypothesis. On the continent of Europe the diagram *plus* the written statement constitutes the hypothesis. Things that are *absolutely obvious* from the diagram are not written unless they are the *essential* conditions of the theorem.

Thus, an American text would say:



Hyp. Two parallel lines AB and CD are intersected by the transversal EF respectively in G and H.

While a German book would state:

Hyp. $AB \parallel CD$.

The German argument is that the diagram shows that EF is a transversal, that it meets AB in G and CD in H, that it is not necessary to say AB and CD are straight lines, since two letters always mean a straight line.

From the viewpoint of science no objection can be made against the American way, but pedagogically the European way is preferable, for it frees the student's mind from a mass of pedantic detail of little value, and *emphasizes the really essential part* of the hypothesis. Thus the student is led to better understanding. It will, for instance, be much easier for him to form a converse when the hypothesis is stated in the brief form, than when in the lengthy form.

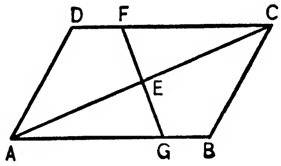
Of course this brevity must not be carried to such an extreme that doubts may arise. Thus lines cannot be assumed to be straight or to be perpendicular merely because they appear so.

Statement of a proof. — The present topic (equal triangles) is very well suited for teaching the conventional form of writing demonstrations, and both here and later exactness of form should be insisted upon just as much as accuracy of thinking.

The form of the following proof may be useful to those who have had no experience in teaching. This proof, however, does not belong to our present chapter, but to a later one.

Theorem. A line drawn through the mid-point of a diagonal of a parallelogram and terminating in two sides of the parallelogram is bisected.

Hyp. $ABCD$ is a $\square$.
AC is bisected at E.
FG is a st. line.
To prove $FE = EG$.



Proof: In $\triangle$s AGE and FEC,	
$AE = EC$.	Hyp.
$\angle EAG = \angle ECF$.	Alt. int. $\angle$s of parallel lines.
$\angle AEG = \angle CEF$.	Vertical angles.
$\therefore \triangle AGE = \triangle FEC$.	($a.s.a. = a.s.a.$)
$\therefore FE = EG$.	Hom. parts of equal $\triangle$s. Q. E. D.

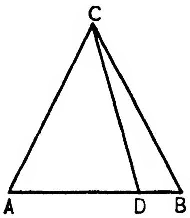
THE LAST THREE PROPOSITIONS OF EQUAL TRIANGLES

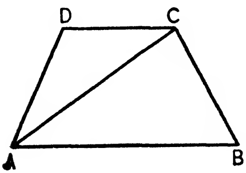
Remarks about the three theorems. — A method of making students discover or analyze the proof of the proposition relating to two mutually equilateral

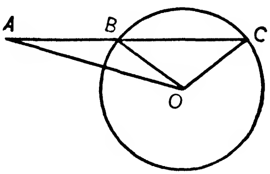
triangles ($s.s.s. = s.s.s.$) was discussed in the preceding chapter.*

The theorem of two triangles that have two angles and a non-included side equal ($s.a.a = s.a.a$) is omitted in quite a number of textbooks. If, however, the solving of exercises is to be made the chief object of geometric study, we cannot dispense with this theorem, and considering how easily it may be proved, it certainly should be studied.

The theorem of two triangles that have two sides and a non-included angle equal respectively is true only under certain restrictions. As students occasionally apply this theorem as if it were generally true, we ought to be able to show them cases in which such a theorem would obviously lead to wrong results. For instance, we should obtain that any line drawn from the vertex of an isosceles triangle divides the figure into two equal triangles, or that a diagonal divides any isosceles trapezoid into two equal triangles. Similarly in the annexed diagram $\triangle ABO$ would equal $\triangle ACO$.







* See Chapter VII, page 101.

Triangles having two sides and a non-included angle equal, however, are equal if the other two non-included angles are not supplementary. A somewhat simpler condition which, however, does not cover as many cases makes the equal angles lie opposite the greater sides.

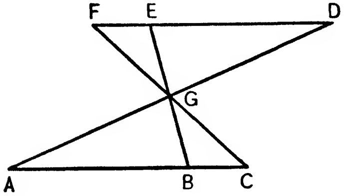
Most textbooks, however, give only the special case in which the angles are right angles, *i.e.* stating that the hypotenuse and an arm of one triangle equal the corresponding parts of another. To make students discover the demonstration of this theorem we had best start with the exercise: The altitude upon the base of an isosceles triangle divides the figure into two equal triangles.

Exercises. — Since the mode of constructing exercises has been fully explained in the preceding paragraphs, we shall not give any simple illustrations of the applications of the present theorems. We shall, however, give a number of more difficult exercises, specially suited to the more ambitious student, which illustrate the following:

METHOD III. **If it is impossible to prove the equality of the required pair of triangles, prove first the equality of some other pair, or pairs, whose homologous parts will enable us to demonstrate the equality of the original pair.**

This principle is very important, and students should become fully familiar with it.

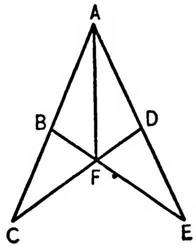
6. *Exercises requiring the equality of several pairs of triangles.*



Ex. 91

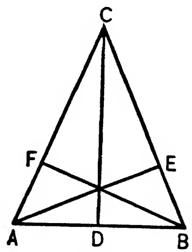
Ex. 91. If $AG = GD$, $CG = GF$, and all lines are straight lines, then $BG = GE$.

Ex. 92. If in polygon $ABCDEF$, $AB = DE$, $BC = EF$, $DC = FA$, and $A \| B\ DE$, then $\angle F = \angle C$.



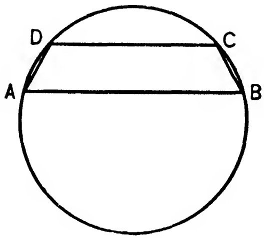
Ex. 93

Ex. 93. If $AB = AD$, and $AC = AE$, then $\angle BAF = \angle DAF$.



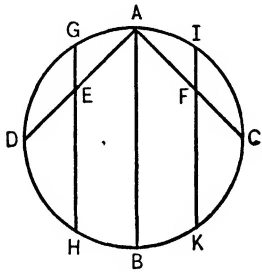
Ex. 94

Ex. 94. If $\angle A = \angle B$, and $AF = BE$, then $AD = DB$.



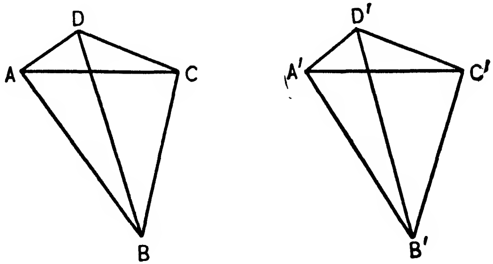
Ex. 95

Ex. 95. If $AB \parallel CD$, then $AD = BC$.



Ex. 96

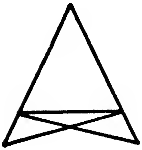
Ex. 96. If $AD = AC$, $DE = CF$, $GH \parallel AB \parallel IK$, and AB is a diameter, then $GH = IK$.

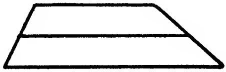


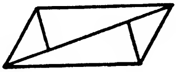
Ex. 97

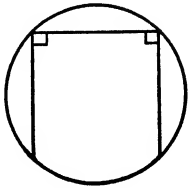
Ex. 97. If in quadrilaterals $ABCD$, and $A'B'C'D'$, $AB = A'B'$, $BC = B'C'$, $CD = C'D'$, $DA = D'A'$, and $AC = A'C'$, then $BD = B'D'$.

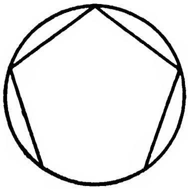
A few additional diagrams will facilitate the construction of more exercises of this type.

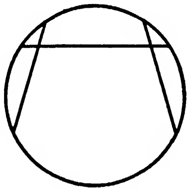








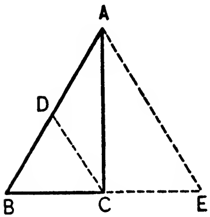




Method for proving the perpendicularity of two lines.

METHOD IV. **To prove that an angle is a right one we usually demonstrate that it is equal to its supplementary adjacent angle.**

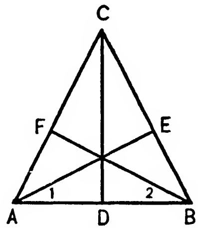
For instance, to prove that triangle ABC is a right one if the median CD equals one half of BA we produce BC by its own length to E and prove the equality of $\triangle ABC$ and AEC. This is easily done if we consider that $CD = \frac{1}{2} AE$, since it joins the mid-points of AB and EB. This illustration requires a theorem which the student at this stage of the work does not know, but his knowledge is sufficient for the following exercises.



Ex. 98. The median to the base of an isosceles triangle is perpendicular to the base.

Ex. 99. The bisector of the vertical angle in an isosceles triangle is perpendicular to the base.

Ex. 100. If in quadrilateral $ABCD$, $AB = BC$, and $\angle A = \angle C$, then $AC \perp BD$.



Exs. 101–104

Ex. 101. If $\angle A = \angle B$, and $\angle 1 = \angle 2$, then $CD \perp AB$.

Ex. 102. CD is perpendicular to AB, if $AE = BF$, and $\angle 1 = \angle 2$.

Ex. 103. CD is perpendicular to AB, if $\angle A = \angle B$, and AE and BF are medians.

Ex. 104. CD is perpendicular to AB, if $\angle 1 = \angle 2$, and AE and BF are altitudes.

CHAPTER IX

PARALLEL LINES

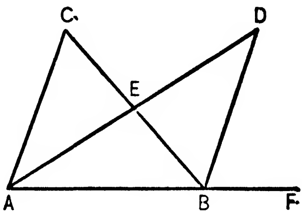
PROPOSITIONS ESTABLISHING PARALLEL LINES

Definition of parallel lines. — The definition that is most widely accepted is: Two lines are parallel if, lying in the same plane, they do not meet, however far produced. Various attempts, however, have been made to formulate other definitions which result in simpler proofs of the theorems. Of these, the two most widely used define parallels as lines that have the same direction, or lines that are everywhere equally distant. Both definitions, while they simplify the proofs of the parallel theorems greatly, are objectionable from the scientific point of view. The first employs the objectionable term direction, the other is so obviously redundant that it has to be rejected. Still, occasionally, a new defender of one of these definitions arises, claiming that he has discovered a new definition of parallel which will simplify the proofs greatly.

Demonstration of the fundamental theorem. — Any arrangement of propositions that exists and that is exact contains one or several difficult proofs. Many American books consider first propositions relating to a perpendicular transversal and obtain then comparatively simple proofs of the general theorems, but the difficulty

is not avoided, it is simply pushed back to the propositions relating to the perpendicular transversal. Moreover, we obtain an unnecessarily large number of theorems.

If we insist upon a rigorous demonstration, it seems that the Euclidean way has not been improved upon. It is true that Euclid presupposes the knowledge of a theorem that is not quite easy to prove, viz. an exterior angle of a triangle is greater than either remote interior angle. But this theorem is not so difficult, if we make it one of a series of originals. Considering two triangles formed by two lines AD and CB that bisect each other, we find $\angle EBD = \angle C$. Drawing now the straight line ABF, we ask whether $\angle C$ or



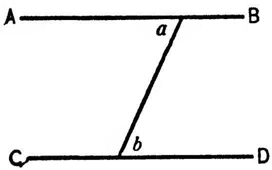
$\angle CBF$ is greater. Erasing the lines AD and BD, we ask whether $\angle CBF$ is still greater than $\angle C$, and finally draw a new diagram ($\angle ABC$ and AB produced) and require a reconstruction of the entire proof.

After the preceding proposition is firmly impressed upon the student's mind by numerous applications, the fact that equal alternate interior angles make lines parallel can easily be obtained by means of numerical questions. Let a transversal form the alternate interior angles a and b, with two intersecting lines, and ask whether or not it is possible that

$$\angle a = 50^\circ, \ \angle b = 60^\circ;$$

or $$\angle a = 50^\circ \ \ \angle b = 50^\circ.$$

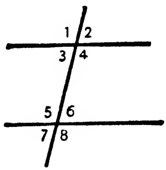
Referring to the next diagram, ask whether the prolongations of AB and CD (*i.e.* towards the right side can meet if $a = 60°$, $b = 50°$;
or $a = 40°$, $b = 50°$;
or $a = 50°$, $b = 50°$.



Discover whether BA and DC produced (*i.e.* towards the left) meet if
$a = 60°$, $b = 50°$;
or $a = 50°$, $b = 50°$.
Can the lines meet at all if
$a = 50°$, $b = 50°$;
or if $a = 60°$, $b = 60°$?

Hence, the lines are parallel for these particular sets of numbers, and it is an easy matter to show that this result is generally true.

Exercises. — The simplest exercises which illustrate the preceding theorem are of course numerical ones. Assign numerical values to any two independent angles of the complete diagram, and ask whether the lines are parallel. Then request student to prove the parallelism of the two lines if certain angles are equal or supplementary; *e.g.* $\angle 2 = \angle 6$, $\angle 7 = \angle 3$, $\angle 1 = \angle 8$, $\angle 7 = \angle 3$, $\angle 3$ is the supplement of $\angle 5$, $\angle 7$ is the supplement of $\angle 4$, etc.



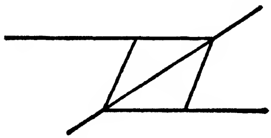
In every case the student must try to discover the equality of a pair of alternate interior angles, and the principal result of work of this kind should be the knowledge of

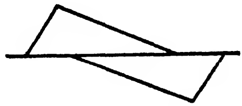
METHOD V. **Parallelism of lines is usually proved by means of equal alternate interior angles.**

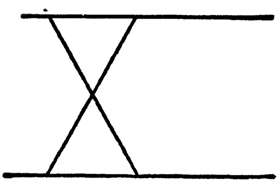
A knowledge of this principle is most essential, and the analysis of any theorem stating parallelism of lines should be commenced with the question: *What is the usual means of proving two lines parallel?*

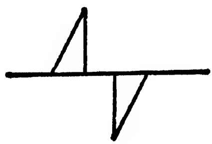
Of course sometimes corresponding angles, or angles on the same side of the transversal, are employed, but the above principle gives the most important method.

While in the simpler exercises, we use vertical angles, supplements of equals, sums of equals, etc., to make alternate interior angles equal, in more complex examples we accomplish the same by means of equal triangles. Below are a few diagrams, each of which may be used for quite a number of originals requiring the demonstration of parallelism.









CONVERSES

General laws. — To discuss the converses of the parallel theorems, a few remarks about converses in general may

be helpful. Usually it is said that the exchange of hypothesis and conclusion produces the converse of a theorem.* Or if the theorem be represented by:

If A is B, then a is b,

its converse would be:

If a is b, then A is B.

The converse of a valid theorem is not necessarily true, a fact illustrated by the converses of the following propositions:

1. If a polygon is regular, a circle can be circumscribed about it.

2. If two parallelograms have equal bases and altitudes, they are equivalent.

3. The diagonals of a rhombus are perpendicular to each other.

4. If a body of gas is compressed, it becomes hotter.

Relation between converse, opposite, and converse of the opposite.—If the theorem be represented by:

If A is B, then a is b;

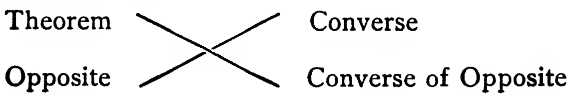
its opposite would be:

If A is not B, then a is not b;

and the converse of the opposite:

If a is not b, then A is not B.

These four theorems are connected in such a way that the validity of one of them always establishes the validity of another one.



* This is not quite exact, see p. 146.

In the preceding arrangement the theorems are connected diagonally, *i.e.*:

If the theorem is true, then the converse of the opposite is true.

If the opposite is true, the converse is true.

If the converse is true, the opposite is true.

If the converse of the opposite is true, the theorem is true.*

These connections are easily proved by the indirect method.

These general matters will become clearer by a concrete illustration. Let a and b be two alternate interior angles formed by two lines AB and CD, and a transversal.

I. THEOREM	II. CONVERSE
If $\angle a = \angle b$, Then $AB \parallel CD$.	If $AB \parallel CD$, Then $\angle a = \angle b$.
III. OPPOSITE	**IV. CONVERSE OF OPPOSITE**
If $\angle a \neq \angle b$,† Then AB is not $\parallel CD$.	If AB is not $\parallel CD$, Then $\angle a \neq \angle b$.

To establish all four theorems, we have to prove only one of the following pairs: I and II, or I and III, or II and IV, or III and IV.

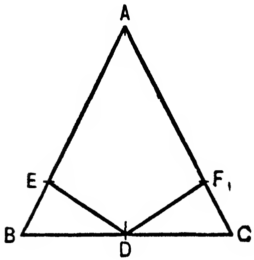
Obviously a knowledge of these relations is a great

* Some textbooks state that both the theorem and its opposite must be proved to establish the converse. This is erroneous; the opposite alone establishes the converse.

† The symbol $\neq$ means "is not equal to."

help in geometric work, *e.g.* if we find the converse of a theorem too difficult, we may attack the opposite instead, etc.

Case of several converses. — If the hypothesis of a theorem consists of several statements, each of them may be exchanged with the conclusion, thus giving rise to several converses, that may be, or may not be, true. For instance:



If in triangle ABC

1. $AB = AC$,
2. $BD = DC$,
3. $DE \perp AB$, $DF \perp AC$,

then $DE = DF$.

Each condition of the hypothesis may be exchanged with the conclusion, thus producing three converses, of which the first two are true, while the third is not.

Law of converses. — If the following three theorems are true, their converses must be true:

1. If $A > B$, then $a > b$.
2. If $A = B$, then $a = b$.
3. If $A < B$, then $a < b$.*

The truth of this assertion can easily be shown by applying the indirect method. Of course the same

* Or more generally: If A may be A_1 or A_2 or A_3 . . . A_n, and this includes all possibilities, and the following propositions are true:

If A is A_1, then a is a_1;
If A is A_2, then a is a_2;
. . . .
If A is A_n, then a is a_n;

then the converses are also true.

fact holds true of the propositions which we obtain by exchanging the first and last conclusion.

Thus, after we prove that equal chords are equidistant from the center, that a chord becomes smaller if its distance from the center increases, and that it becomes larger if its distance from the center decreases, — the converses of these propositions can be accepted as true without further proof.

Pedagogic value of the preceding laws. — While the general logical notions discussed in the preceding paragraphs furnish undoubtedly very useful tools for geometric work and are very valuable to the teacher, it would be a fatal error to teach general logical propositions before the concrete theorems. The student does not appreciate these general facts, nor will he as a rule understand the meaning of such abstract theorems until he is well acquainted with a large number of concrete illustrations. Hence such matters should never come in the beginning of geometric instruction. They may be given in a review course, but even then more as supplementary matter than as an integral part of the regular work.

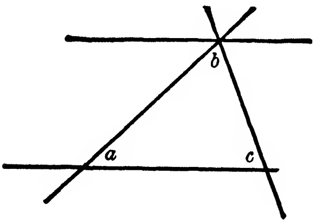
Converses of the theorems on parallels establish the equality of angles, if parallel lines are given. The first proof * of this kind is always rather difficult, and if the students are immature, it is advisable to teach the opposite theorem first, viz.: If two alternate interior angles are unequal, the lines are not parallel. Logically

* It should be noted that these converses depend upon Euclid's postulate (Chapter IV), while the direct theorems do not require it.

the converse would then require no proof at all; for the beginner, however, a proof should be given, which, in this case, is very simple. Such an arrangement removes some difficulties, even though it increases the number of propositions.

Exercises illustrating the converses of the parallel line propositions can easily be formed. They may be classified as follows:

1. *Numerical examples* are adapted for oral work. If two parallels intersected by a transversal are given, assign a numerical or literal value to any angle of the diagram and require the value of all others. Construct two transversals intersecting one of the parallels in a common point, and assign values to any two independent angles and let the students find all other angles.*



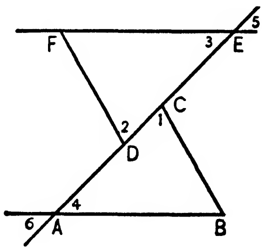
Introduce two parallel transversals, bisectors of certain angles, perpendiculars to a transversal, etc., etc.

2. *Simple general examples.* — In diagrams similar to those of the preceding class, let students prove that certain angles are equal, others supplementary or complementary, that one angle equals the sum of some other two. In other words, establish in a general form the facts which were represented numerically by the exercises of the preceding section.

* Every exercise on pp. 107 and 108, Chapter VI, may be used as the basis of a corresponding exercise relating to the above diagram.

3. *Exercises preparing students for later topics.* — The preceding diagram makes it possible for the student to find the sum of angles *a*, *b*, and *c*, and may be used to lead him directly to the theorem of the sum of the angles of a triangle. Similarly he may find the propositions relating to the exterior angle of a triangle, to opposite angles of a parallelogram, to the sum of the angles of a quadrilateral, etc.

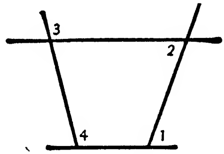
4. *Equality of triangles.* — The number of originals requiring equality of triangles can now be greatly increased, as we have a new means of obtaining equal angles. The diagrams on pages 117 and 122 may be used for the construction of a great many originals, that assume certain pairs of parallel lines and require the equality of certain other lines or angles. Thus, in the annexed diagram, we may say:



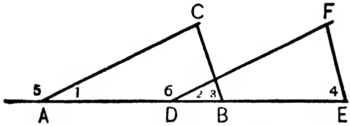
If $AB \parallel EF$, $AB = EF$, and $EC = DA$, then $BC = DF$, etc., etc.

5. *Combination of the direct parallel propositions and their converses.* — We may use the equality of certain angles to prove the parallelism of lines and therefrom again deduce the equality of other lines or angles, or we may start from parallelism, obtain the equality of certain angles and triangles, and finally demonstrate the parallelism of other lines.

The two following exercises illustrate these two methods:



Ex. 1. If $\angle 1 = \angle 2$, then $\angle 3 = \angle 4$.



Ex. 2. If $AC \parallel DF$, $AC = DF$, and $AD = BE$, then $CB \parallel FE$.

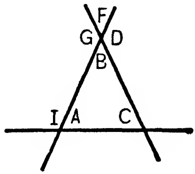
Sum of the angles of a triangle. — Among the various by-products of the preceding methods of solving exercises are the theorem of the angle-sum of a triangle, the theorem of the exterior angle of a triangle, the parallelogram propositions, etc.

Hence students will experience no difficulty when analyzing the theorem of the angle-sum of a triangle. The originals based upon this proposition, however, may deserve a short discussion.

To construct exercises relating to the angles A, B, and C of triangle ABC, we have to consider that the three quantities are always connected by one equation, viz., $A + B + C = 180°$, and that consequently two other (independent) equations will determine the values of the angles.

Since in the annexed diagram all angles are known if A, B, and C are given, it follows that the twelve angles of the diagram can be determined if two equations connecting independent angles are given.

Thus to find all twelve angles, we may assign numerical values to any two independent angles of the diagram, or we may make $\angle A = \angle F$ and $\angle C = 22°$, or we may make $\angle A = 90°$ and $\angle G + \angle C = 140°$, etc.



For more difficult examples we may use more complex equations, as

$$A + 5\,B = 240°, \quad C - B = 30°,$$

or
$$G - A + C = 130°, 2\,D - I = 100°.$$

Of course a great many originals of entirely different character may be based upon this proposition, but it would lead us too far to discuss them.

CHAPTER X

MISCELLANEOUS TOPICS OF THE FIRST BOOK OF GEOMETRY

HYPOTHETICAL CONSTRUCTION AND THE ISOSCELES TRIANGLE

Isosceles triangle. — Euclid proved the equality of the base angles of an isosceles triangle by applying the figure to itself so that each leg coincided with the other. While this is a very ingenious and short method, it is not the kind of proof that appeals to young students. Moreover, it is always preferable to let students apply the usual fundamental method, *i.e.* the method of equal triangles.

In the prevailing arrangement of propositions, the only method that leads to two triangles the equality of which can be proved by the student at this stage, is based upon the bisection of the vertical angle. As the student, however, cannot foresee this result, it would be advisable to let him try to obtain two equal triangles by means of the median or the altitude, in order that he may discover for himself why the bisector alone can be used.

The only point relating to the proof of this proposition which the student has to remember is the fact that the bisector of the vertical angle does lead to the two equal triangles.

Hypothetical construction. — *What is a hypothetical construction?* — In the proof discussed in the preceding paragraph, use has been made of the bisector of an angle, and this is done before the method of constructing such a bisection has been established. This introduction of a line or other figure, before it is shown how it can be constructed, is called a *hypothetical construction.* Euclid never used hypothetical constructions, while practically all modern books use them, more or less.

There are, however, many critics who attack the use of hypothetical constructions. Especially does the use of the bisector in the isosceles triangle seem to invite the criticism of certain writers, who do not appear to realize that there are other cases just as glaring in nearly all textbooks.

Logical aspect of hypothetical constructions. — Is the use of hypothetical constructions justified from the logical point of view? Is it logical to introduce the bisector of an angle, before we possess an exact method of constructing it? Or are the numerous critics justified in denouncing such a practice as illogical?

The answer to these critics is that it does not matter in the least whether the line drawn is the exact bisector or not. The proof of a demonstration does not depend upon the accuracy of its diagram. As soon as we admit that there is a bisector, we can give an absolutely exact proof, and the validity of such a proof does not at all depend upon our ability to draw this line exactly.

Such a use of quantities that we cannot accurately determine is absolutely general in mathematical science. We form relations between the roots of an equation, not only before we have determined them, but even in cases when it is absolutely impossible to determine them. We refer coördinates to the center of gravity of a system, before we know the location of this center, etc. If we excluded any quantity before we had a method of determining it, we would have no higher mathematics.

It is true, however, that we assume something in a hypothetical construction, and this is the *existence* of the line or other figure, *e.g.* the existence of the bisector of an angle.

But there is an enormous difference between the assertion that we have to show that a bisector is possible, and the contention that we have to find the entire method for constructing such a line. Not even the possibility of the above construction has to be proved, but simply the fact that somewhere there is a bisector. If, for instance, a demonstration should require seven points dividing a circumference into seven equal parts, it would be perfectly logical to use the seven points. For, while it is impossible to obtain seven such points by an exact construction, it is a very easy matter to show that such points *do exist*, and the logic of the proof does not depend upon our ability to get an exact diagram.

Hence the flaw in hypothetical constructions is not of the kind that most critics claim, and in nearly all cases of elementary geometry this flaw can be entirely

removed. In the above theorem, for instance, it would be a very simple matter to demonstrate that there is always a bisector, although very few teachers would think it worth while to introduce such a demonstration.

Pedagogic aspect of hypothetical constructions. — The logical objections against hypothetical figures are therefore not of such a kind that we should exclude this method. On the other hand, the introduction of hypothetical diagrams undoubtedly simplifies elementary geometry. The greater simplicity of our modern way of presenting this science as compared to Euclid's is largely due to the use of this method, and hence its use in elementary teaching must be recommended.

It is true, however, that students who have never drawn a perpendicular, nor a parallel, nor a bisector may be somewhat confused by the introduction of such lines, and may ask how these lines can be obtained. The discussions of the preceding paragraph would in such a case be of no help to the students, but would rather confuse them still more. A little time devoted to drawing exercises before demonstrative geometry is taken up will remove all such difficulties.

Applications of the preceding theorem. — Among the many applications of the isosceles triangle proposition that are possible, two classes may be mentioned, viz., the calculation of certain angles and the finding of equal lines and angles.

Since the three angles of an isosceles triangle are connected by two equations, any other independent equation will determine all angles. Hence, by assign-

ing a value to any of the twelve angles of the complete diagram, we may determine the others. Similarly any equation as $A - C = 20°$, $2A + 3B - C = 120°$, etc., will enable us to find all angles.

The direct proposition and its converse may be used to prove the equality of lines and angles, and there are cases which are much more effectively attacked in this manner than by equal triangles. While logically this is only a special case of equal triangles, practically it is a new method. Hence the student had better become familiar with

METHOD VI. **The equality of angles is occasionally proved by the isosceles triangle proposition.**

SIMPLE CONSTRUCTIONS

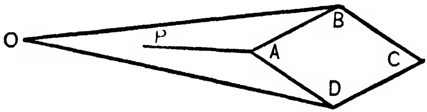
Straightedge and compasses. — All constructions in elementary plane geometry have to be carried out by means of two instruments, viz., the compasses and the straightedge. Of these, only the compasses deserve the name of an instrument, as the straightedge is simply a model of a straight line that enables us to copy a line which was constructed by some one else. It is not any more an instrument than a piece of board that has been given the form of an ellipse or a parabola, and that may be used to draw such curves at the blackboard. Since the perfect rectilinear motion has a certain value in mechanics, an instrument accomplishing this was sought for a long time. Watt's parallelogram, used on his steam engines, produces an approximately

rectilinear motion, but Peaucellier's linkage * was the first instrument to accomplish this exactly.

It should be borne in mind that the restriction to rules and compasses is purely conventional, due to the great simplicity of these instruments, and not to any intrinsic qualities of geometric figures.

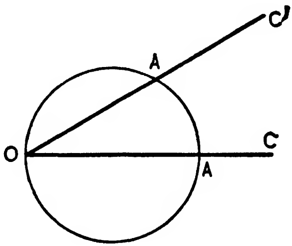
Attempts have been made to effect the constructions of geometry by means of the compasses alone, or by means of the straightedge alone. By the last method, however, only a limited number of problems † can be

* Peaucellier's linkage consists of 7 rigid links, connected by the joints A, B, C, D, O, and P, of which only two, O and P, have fixed positions. $PO = PA$, $DO = BO$, $AB = BC = CD = DA$. When the instrument swings about O and P, the point A obviously moves in a circle, and it can be proved that C describes a straight line.



The proof depends upon the following facts:

1. $OA \times OC = \overline{OD}^2 - DA^2$, *i.e.* a constant.



2. If $OA \times OC$ = a constant, and A moves in a circle, then C moves in a straight line.

† If the unknown quantity, expressed algebraically, is rational, the ruler alone will effect a construction.

solved. On the other hand, Mascheroni, in 1797, succeeded in showing that all constructions that can be effected by ruler and compasses can also be accomplished by the compasses alone. Later on, Poncelet demonstrated that the same problems can also be solved by the use of *one* fixed circle, and the use of the straight-edge only.

Pedagogic remarks. — It is very desirable that the simple constructions — *i.e.* drawing of bisector, perpendicular, parallel, etc. — should occur as early as possible in the course, for this will do away with further "hypothetical constructions," and besides, these constructions are of great simplicity and interest to the beginner. Since several of these constructions depend upon the third proposition of equal triangles (*s.s.s.* = *s.s.s.*), they have to be placed later than this proposition. It is, however, a mistake to place them at the end of Book I, or even of II, as some authors have done. The theory of these constructions offers little difficulty, especially if we give a large number of applications.

One feature that deserves special attention is the insistence upon accuracy of language and drawing. While the diagram of a theorem has nothing to do with the validity of the proposition, the diagram of a problem is the essential part of the work. Moreover, continual free-hand construction will sometimes lead to loose thinking. Students will talk glibly about an arc drawn from O as a center, and will at the same time draw an arc whose prolongation passes through O. Accurate drawing makes such matters impossible, and

hence — in the beginning at least — all constructions should be effected with ruler and compasses.

Accuracy of expression is most essential for brief and concise descriptions of the constructions. It may be easily obtained by making students know exactly the few typical phrases that occur in such work. "From A as a center with a radius equal to CD draw an arc." "On AB lay off $AC = MN$." "From A draw $AB \perp MN$." "Through O draw $AB \parallel CD$," etc., etc.

Insist also upon designating a new line, etc., — if it is going to be designated at all, — as soon as it is introduced. Do not say: "Through A draw a perpendicular to MN, and designate it by AB," but, "Through A draw $AB \perp MN$."

Do not give details of preceding constructions. For instance, if any construction makes use of the bisecting of an angle A, it is sufficient to state "bisect angle A," although in the beginning we may require the student to *draw* all details of this construction. Or if a construction requires the transformation of a rectangle $ABCD$ into a square, and this problem has been studied before, state only: "Transform $ABCD$ into a square."

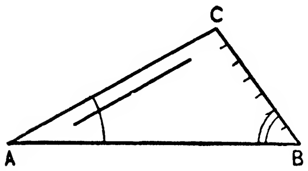
In complex examples make a graphic distinction between given lines, lines of construction, and resulting lines by using different colors or by drawing one kind of line thin, the other dotted, and the third heavy.

UNEQUAL LINES AND ANGLES

Proofs of the propositions. — The first of the propositions establishing the inequality of lines compares the

sum of two sides of a triangle with the third one, and is so simple that it hardly requires any comment. The second, however, viz., that in a triangle the angle opposite the greater side is the greater one is difficult to analyze. The writer employs paper folding for this purpose, which usually leads students to the discovery of a proof.

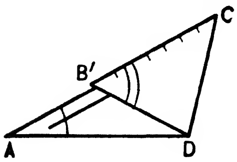
Cut out a piece of paper and mark, on both sides of the paper, the sides whose lengths we assume to be different. Mark also differently, on both sides of the paper, the angles whose relative size we wish to discover, and ask the following questions:

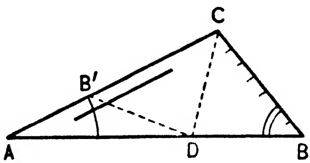


Which is the only theorem which tells that one angle is greater than another? *Ans.* The exterior angle of a triangle, etc.

Who can fold this paper so that B becomes the exterior angle of a triangle, while A becomes a remote interior?

Let all students cut out pieces of paper and try, and soon some will find the answer that is indicated in the annexed diagram. Students *see* now the truth of the

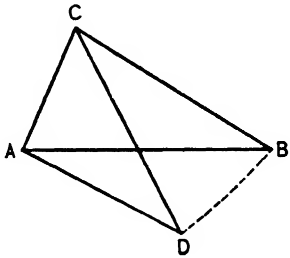


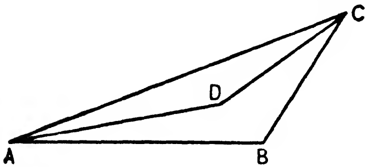


proposition, and it remains to translate this idea into geometric terminology. We unfold the paper, find that

CD is the bisector of $\angle C$, find the position of B', the equality of $\triangle BDC$ and $B'DC$, etc.

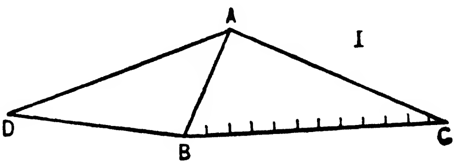
The third proposition, which compares two triangles that have two sides respectively equal but the included angles unequal, has a great many proofs, all of which require the superposition of two equal sides. One proof, for instance, which is not so very difficult to discover, places the triangles in the position indicated in the diagram. If we wish to prove that $AB > AD$, we would naturally inquire into the relative sizes of angles ABD and ADB. But as the base angles of isosceles triangle CDB are equal, the proof is easily completed. This proof, however, has a drawback which is peculiar to many demonstrations of this proposition, viz., it has to be modified considerably for different figures, *e.g.* for the annexed diagram.



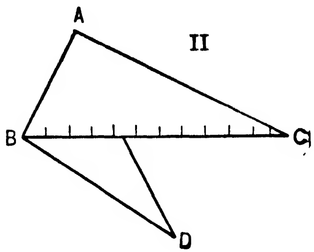


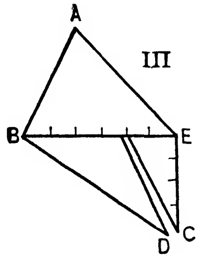
The proof which is usually given in textbooks has not this disadvantage, although its analysis is rather difficult. It may, however, also be discovered by paper folding.

Cut out a piece of paper as indicated in diagram I, and let BC and DB be the lines



whose inequality has to be proven. Fold the paper over as in diagram II, and require now the students to fold





it so that BD forms one side of the triangle, while BC forms the other two sides. This will lead to figure III and the entire proof.

The converses of these cases require, according to the law discussed on page 146, no demonstrations. While we would not make use of this fact, it is well to bear in mind that in all such cases the indirect method will effect a proof.

Simple application of the propositions. — The analysis of an original requiring the inequality of lines should be started by the question: "What means do we have to prove the inequality of lines?" The only means which we have so far are the three theorems, to which we shall refer in the following as (1), (2), and (3) respectively. Number (1) has to be used when no relation of angles is given or can be found. Number (2) may be used if the lines which we wish to compare are sides of *one* triangle and some facts relating to angles can be found. Number (3) has to be used if the two lines whose inequality

we have to prove lie in different triangles and some facts relating to angles may be found.

Of course this is not absolutely definite, but this is one of the peculiarities — or shall we say charms? — of geometry. The work cannot be done without some thinking, some originality, on the part of the student.

Before actually demonstrating such originals, it is advisable to let students decide in a large number of cases which of the above methods is most likely to lead to a result, although it must be borne in mind that sometimes two or even three methods may be used effectively.

The given data may be indicated graphically, *e.g.* one color always representing the larger, another color the smaller side or angle, as the case may be. A few exercises adapted to such a choosing of the proper means are the following:

Ex. 1. If AB and CD are two intersecting lines, $AB + CD < AD + CB$.

Ex. 2. In triangle ABC if $AB = BC$ and D lies in AB, prove that $DC > DA$.

Ex. 3. If two sides of a parallelogram are unequal, the angles formed by the diagonals are unequal.

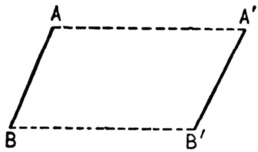
Ex. 4. A point without the perpendicular bisector of a line is unequally distant from the ends of the line.

The applications of the converses are analogous to those of the direct propositions and hence hardly require a discussion.

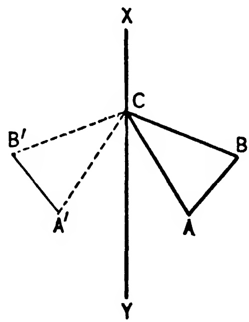
Difficult originals relating to unequal lines. — The methods of the present paragraph may be helpful to the teacher, while in general they are not adapted for

classroom work in a high school. These methods derive a particular value from the fact that similar ones are frequently used for attacking problems.* In hard examples the difficulty is usually due to the circumstance that the lines to be compared do not lie together, *i.e.* do not form one triangle or two triangles. In such cases it is necessary to move some parts so as to bring certain lines or angles together. In geometry such modes of moving figures are frequently used, and the most important ones are:

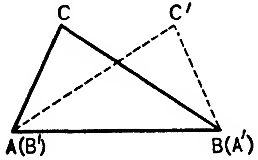
1. *Translation.* — A figure is subjected to a translation if all its points describe equal and parallel lines. Thus AB may be translated into the position $A'B'$. Of course $A'B'$ would also be equal and parallel to AB.



2. *Turning about an axis.* — If $\triangle ABC$ is turned over about the axis xy, it will take the position $A'B'C$. ABC is symmetric to $A'B'C$ with respect to the axis xy.



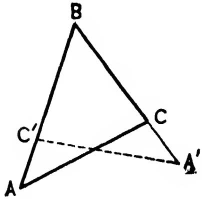
A *special* case of the preceding method is the *turning over a figure so that one of its lines, e.g. AB, coincides with BA.* If we apply AB to BA so that A occupies the position formerly held by

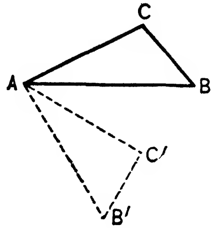


* See Chapter XV.

B, and B the position formerly held by A, then the $\triangle ABC$ will assume the position $A'B'C$.

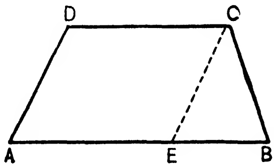
Similarly, we may apply an angle ABC to CBA. The resulting triangle is $A'B'C'$.





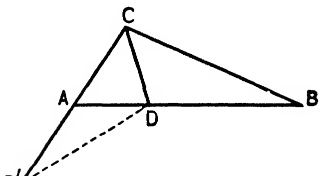
3. *Rotation about a point.* — In the diagram $\triangle ABC$ is rotated about A as center through an angle of 60°.

The use of these methods is explained by the following outlines of four propositions:



1. If in trapezoid $ABCD$, $DA > CB$, then $\angle B > \angle A$.

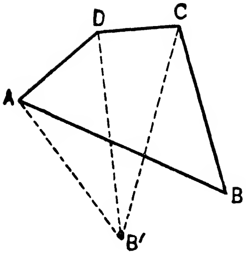
Translate DA into the position CE (*i.e.* draw $CE \parallel DA$), then $CE > CB$, $\therefore \angle B > \angle CEB$, $\therefore \angle B > \angle A$.



2. In $\triangle ABC$, $BC > AC$, and CD bisects $\angle C$.

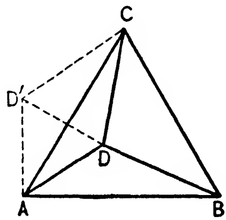
To prove $DB > AD$.

Turn $\triangle CDB$ over about CD, thus taking the position CDB'. To prove that $DB' > AD$, we have to compare $\measuredangle B'$ and $B'AD$, or (since $B = B'$) $\measuredangle B$ and $B'AD$. Obviously the latter, being an exterior angle of $\triangle ABC$, is the greater one. Hence the proof follows easily.



If in quadrilateral $ABCD$, $AB > CD$, and $BC > DA$, prove that $\angle D > \angle B$.

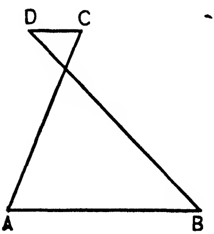
Since no two of the unequal sides lie together, we turn $\triangle ABC$ over so that AC (not drawn in diagram) coincides with CA, and the triangle takes the position $A'B'C$. Drawing now $B'D$, we obtain two triangles $AB'D$ and $DB'C$, each containing two unequal sides. Adding the unequal angles obtained herefrom, we get $\angle D > \angle B'$ or $\angle D > \angle B$.



3. If a point D be taken in equilateral triangle such that $\angle ADB > \angle ADC$, then $DC > DB$.

Rotate $\triangle ADB$ about A until it takes the position $AD'C$. We have now to compare CD and CD', which suggests the drawing of $D'D$ and the comparison of the angles $CD'D$ and CDD'. As $\triangle ADD'$ is isosceles, the conclusion is easily obtained.

EXERCISES



Ex. 1. If $AB \parallel CD$ and $DB > CA$, then $\angle A > \angle B$.

Ex. 2. If in triangle ABC the median CD is drawn, and $BC > CAE$, then $\angle ACD > \angle BCD$. (Rotate $\triangle DAC$ about D through 180°, or translate CA.)

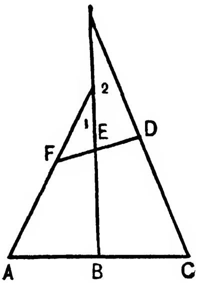
Ex. 3. If in quadrilateral $ABCD$, $AB > DC$, and $AD = BC$, then $\angle C > \angle A$ and $\angle B > \angle D$.

Ex. 4. If in quadrilateral $ABCD$, $AB > CD$ and $\angle B = \angle D$, then $BC > DA$.

Ex. 5. If a point E within parallelogram $ABCD$ be joined to the four vertices, and $EA > EC$, $ED > EB$, then $\angle CEB > \angle DEA$.

Ex. 6. In the same diagram, if $EA > EC$ and $ED = EB$, then $\angle CEB > \angle DEA$.

Ex. 7. If point E within a square $ABCD$ be joined to A, B, and C, and $AE > CE$, then $\angle BEC > \angle BEA$.

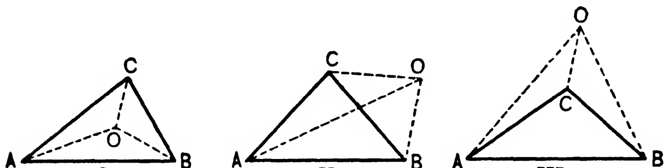


Ex. 8. If in the annexed diagram $AB = BC$, $FE = ED$, and $DC > FA$, then $\angle 1 > \angle 2$. (Compare Ex. 4, p. 240.)

POLYGONS

Positive and negative quantities in geometry. — In recent years certain writers on mathematical pedagogy have declared with great emphasis that it was of the utmost importance that polygons and angles should be lettered counter-clockwise. The reason for such a usage is due to modern geometry. Modern geometry considers the *algebraic* values of geometric quantities, *i.e.* it makes use of negative lines, negative angles, and negative areas, and by this method greatly simplifies many statements. While elementary Euclidean geometry frequently has to modify its statements for different figures, or in other words has to distinguish between several "cases," modern geometry covers all such cases by one statement. To illustrate by a concrete example,

let us join any point O to the vertices of a triangle ABC, and let us discover the relation between $\triangle ABC$ and the triangles OAB, OBC, and OCA. Elementary geometry has to discriminate between seven different cases, three of which are illustrated by the following diagrams.



I II III

In these diagrams we have respectively:

$$\text{I.}\quad \triangle ABC = \triangle OAB + \triangle OBC + \triangle OCA.$$
$$\text{II.}\quad \triangle ABC = \triangle OAB - \triangle OBC + \triangle OCA.$$
$$\text{III.}\quad \triangle ABC = \triangle OAB - \triangle OBC - \triangle OCA.$$

In modern geometry, however, a clockwise sequence of the letters denoting a triangle represents a negative area,* hence $\triangle OBC$ in diagrams II and III is a negative quantity, and similarly $\triangle OCA$ in diagram III. Hence we have for all possible figures

$$\triangle ABC = \triangle OAB + \triangle OBC + \triangle OCA.$$

Similarly, by considering angles read counter-clockwise as positive, those read clockwise as negative, we

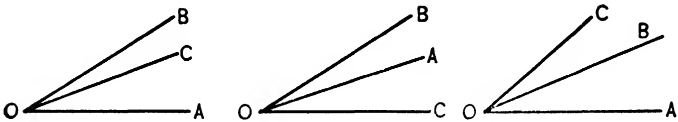
* The reason for selecting the counter-clockwise sequence as positive is due to the fact that the determinant,

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

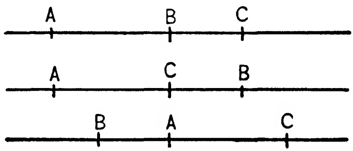
which gives the area of a triangle whose vertices are x_1, y_1; x_2, y_2; and x_3, y_3, is positive if the vertices 1, 2, and 3 follow in counter-clockwise order.

may in the following diagrams state quite generally, no matter what the positions of A, B, and C are:

$$\angle AOB + \angle BOC = \angle AOC.$$



Using directed lines, we can assert quite generally with reference to three points A, B, C lying in a straight line: $AB + BC = AC$; or, $AB + BC + CA = 0$. These statements do not depend upon the positions of A, B, and C.*



The great usefulness of such conventions to advanced geometry cannot be denied, but this does not decide the question, whether or not we should insist upon these matters in a secondary school. Not one high school student in a thousand will ever study modern geometry, and the few who do will grasp these ideas in a few minutes. Hence the *usefulness* of these conventions does not form a sufficient reason for introducing them in our schools.

"But these matters are so exceedingly simple," we are told, "that students will learn them without any extra effort whatsoever." In some cases this is true, but in others it would mean an additional, and quite unnecessary, burden put upon the pupil. Take the case

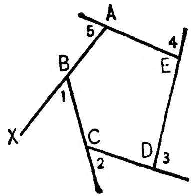
* If we admit imaginary quantities, then $AB + BC + CA = 0$, even if C does not lie on AB.

of a somewhat complex diagram, *e.g.* a triangle with its altitudes and the lines joining the feet of the altitudes, and force the student, who is reciting a difficult demonstration, to read every angle mentioned counter-clockwise, and there is no doubt that you add thereby a very decided difficulty. This difficulty is utterly uncalled for, as it has nothing whatever to do with the demonstration, and it distracts the student's mind from the true issue. Hence it seems absurd to insist upon a rigid application of these modern conventions. True, in very simple examples, as in lettering a triangle or a quadrilateral, no harm would be done by counter-clockwise notation, and it may even be recommended in such cases, but solely for the sake of uniformity.

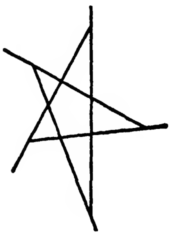
To sum up the whole situation: Nothing can be said against the use of counter-clockwise notation in very simple examples, but the importance of the entire matter has been greatly overrated by some of our pedagogues.

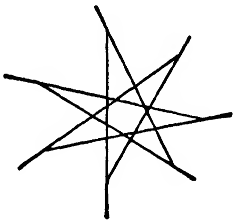
Remarks on two theorems. — The sum of the angles of a polygon, when expressed in right angles, is sometimes stated in an ambiguous way, that may leave the reader in doubt, whether this sum is $2n-2$ or $2(n-2)$ right angles. The use of straight angles, or of the algebraic symbols $2(n-2)$ will obviate this difficulty. In giving the usual demonstration of this proposition, some students are inclined to add triangles instead of angles of these triangles. We may test the real understanding of the students by joining a point within the polygon to all the vertices, and finding out whether they will add the triangles in this case, also.

The truth of the proposition of the sum of the exterior angles can be shown in a concrete way by placing a pencil in the position BX and rotating it by the amount of $\angle$ 1, moving it then to next vertex, adding a rotation equal to $\angle$ 2, and so forth. It is obvious that the pencil when returning to its original position has rotated through an angle of 360°, *i.e.* the sum of the exterior angles is equal to 4 rt. $\measuredangle$s.



This method can also be applied to exterior angles of star-shaped polygons, resulting in the first figure in





2 × 360°, in the second, in 3 × 360°. It is then an easy matter to obtain the sum of the interior angles of such star-shaped polygons.

Exercises referring to polygons. — Oral exercises are well adapted for familiarizing students with the simpler applications of these propositions: Find the sum of various polygons in right angles, straight angles, and degrees; find the number of sides of polygons, the sums of whose angles are given in right angles, straight angles, or degrees; find each angle of equiangular polygons, etc.

For more difficult examples, it may be well to point out that the exterior angle proposition frequently can be employed to greater advantage than the sum of the interior angles. To find, for instance, the number of sides of a polygon each of whose interior angles is 170°, it is not advisable to give the algebraic solution based upon the equation $\frac{n-2}{n}=\frac{170}{180}$, but to consider that each exterior angle is 10°, and that hence the number of such angles, and the number of sides, is 36.

CHAPTER XI

METHODS OF ATTACKING THEOREMS

Typical methods of proving geometric facts. — The fundamental idea of the analysis of geometric theorems was explained in Chapter III. There it was shown that every analysis starts by examining the various means by which the proposition may be proved. A knowledge of the most generally used "means" of proving certain geometric facts is therefore of the utmost importance. Every proposition, axiom, or definition may be used as such a means, but far more important are the general typical "methods," such as the methods for proving the equality of lines, the parallelism of lines, etc.

A thorough acquaintance with these "methods" is indispensable for successful work in attacking problems; and it should be one of the chief aims of geometric instruction to familiarize the students with them.

To the six methods previously given two more may be added: *

METHOD VII. **To prove that a line is twice as large as another we usually double the smaller, and prove that its double equals the greater, or sometimes we bisect the greater, and prove that its half equals the smaller. The same relation between angles is proved in a similar way.**

* These eight methods refer to the first book of geometry. For methods relating to the other books of geometry, see Schultze and Sevenoak's Geometry.

METHOD VIII. **To prove that the sum of two lines *a* and *b* equals a third line *c*, construct the sum of *a* and *b* and prove that it equals *c*, or construct the difference between *c* and *a* and prove that it equals *b*.**

The question that should always open the attack of a problem should be: What means have we of establishing this conclusion? In the absence of general methods, we have to look for propositions, or even axioms and definitions.

Every proposition in the textbook and every original should be so attacked, *i.e.* analyzed. Thus in teaching the proposition, "If the opposite sides of a quadrilateral are equal, the figure is a parallelogram," we should ask, What is the usual method of demonstrating the parallelism of lines? And after the original question is in this manner reduced to the equality of two angles, we should ask for the method of demonstrating the equality of angles, etc.

Analysis. — As explained in the preceding paragraph, the student at this stage of the work should be thoroughly familiar with the practical use of simple analysis, even though the teacher may not have found it necessary to refer explicitly to the term analysis or to discuss its peculiarities in detail. Before Book I is completed, however, this should be done. All that was previously mentioned about this method should be summarized, and its application to more difficult theorems should be studied.

In making a complete analysis we should study as far as possible *every* means that we have at our dis-

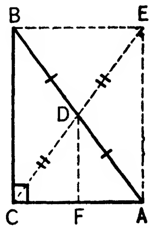
posal. This will reduce the original theorem to a number of others, any one of which may lead to a solution. The same method is to be applied to each of the new questions, and it can easily be seen that in some cases a great number of proofs may result. In general some of these proofs are simpler than others, and the skill of the student will appear in a wise selection of the means.

At the start, and whenever the original question is reduced to another, the student should survey the diagram and determine all known facts. Thus, he should find out what angles or lines are equal, what numerical values of angles are known, etc., and he should represent these facts graphically in the diagram.

Not every theorem can be analyzed in an absolutely stereotyped manner; the more difficult require a certain amount of ingenuity. But a study of such analytic methods will greatly enlarge the power of the weak as well as of the strong student.

A few illustrations may be useful.*

1. *Theorem.*—The median CD to the hypotenuse AB of a right triangle is one half the hypotenuse.

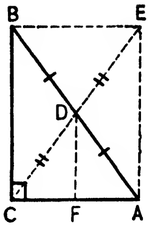


Analysis.—According to one of the above methods, either (a) double CD, or (b) bisect BA.

(a) Produce CD by its own length to E and prove that $CE = AB$.

Survey the diagram: $AD = DB$, $CD = DE$, which shows $AEBC$ is a parallelogram; $\angle C = 90°$,

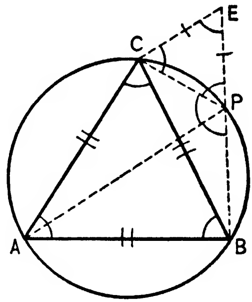
* For further simple illustrations see Schultze and Sevenoak's Geometry.



since $AEBC$ is a rectangle. The equality of CE and AB follows from equal triangles, or from the proposition relating to the diagonals of a rectangle.

(*b*) Since D is the midpoint of AB, we have to prove that $CD = DA$. The means for proving the equality of lines is usually a pair of equal triangles, and if there are no triangles, construct a pair; hence draw $DF \parallel BC$, and find equal parts.

2. *Theorem.** — If ABC is an equilateral triangle inscribed in a circle and P is a point in arc BC, then $PA = PB + PC$.



Analysis. — We have two means of showing this conclusion, viz., either

I. Draw the sum of PB and PC, or

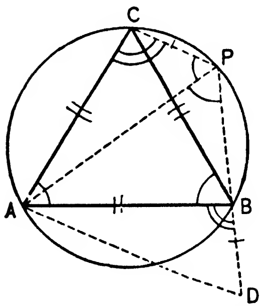
II. Draw the difference of PA and PC.

I. We may draw the sum of PB and PC by either

(*a*) prolonging BP, or

(*b*) prolonging PB.

(*a*) Produce BP to E, so that $PE = PC$, and prove the equality of BE and PA. Survey the diagram. The following angles are known to be 60°: A, ACB, ABC, APB, APC, CPE.



$PC = PE$ and hence $\angle E = \angle ECP = 60°$. Therefore $CE = CP$.

The equality of lines depends usually upon a pair of equal triangles. The triangles are CAP and CBE, and their equality is easily established.

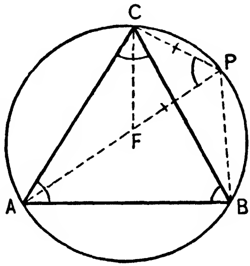
(*b*) Produce PB to D, so that $BD = CP$, and prove $PA = PD$.

Here we cannot find equal triangles, but a careful survey of the diagram will lead to a solution, which, however, is more difficult than I, (*a*).

* This analysis requires a knowledge of Book II.

Survey the diagram and mark all angles of 60° by one arc. As $\angle APD = 60°$, $\triangle APD$ has to be equilateral if the theorem is true. Hence, if we could prove $AD = AP$, the theorem would be established.

The equality of AD and AP follows from the equality of the triangles ACP and ABD.



II. The difference between PA and PC may be constructed by laying off PC either

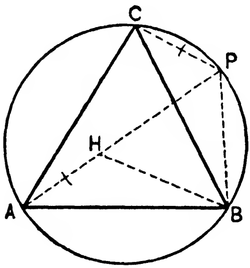
(*a*) on PA, or
(*b*) on AP.

(*a*) On PA lay off $PF = PC$ and attempt to prove: $AF = PB$

Survey the diagram:

$$\angle A = \angle ABC = \angle BCA = \angle APC = \angle BPA = 60°.$$

Since $PC = PF$, $\angle PCF = \angle PFC = 60°$, hence $CF = CP$.



The means for proving $AF = PB$ are the triangles AFC and BPC, whose equality is easily established.

(*b*) On AP lay off $AH = CP$ and prove $HP = PB$. Since we have no equal triangles, containing HP and PB as homologous parts, we try to show that PHB is equilateral, or that $HB = PB$, a fact that follows from the equal triangles ABH and CBP.

The indirect method. — A proposition which denies another one is called its contradictory. Thus, if we consider the proposition,

If A is B, then a is b,

its contradictory would be,

If A is B, then a is not b.

Instead of proving that a theorem is true, we may show that its contradictory is absurd. Such a method of

demonstration is called "reductio ad absurdum" or the indirect method. More concretely we may explain this method as follows:

If, in general, A must be either B or C, or D, and we wish to prove that under the particular conditions of the hypothesis A must be B, we may show this either directly, or by demonstrating that the conclusions A is C and A is D lead to contradictions. These may be contradictions of the hypothesis, or of a previously proven theorem, or of an axiom.

In regard to the application of the indirect method no general rules can be given. It may be tried by the students whenever other methods fail. It is frequently applied for the proofs of converses, and can always be applied in case of the law of converses that was discussed in Chapter X.

Attacking a theorem as a problem. — Most theorems may be attacked as problems.* Thus, instead of proving the proposition of the square of a side opposite an acute angle, we may require to find the side of a triangle opposite an acute angle whose other two sides are respectively equal to b and c, if the proper projection is p. In the study of Book I, this may be applied to theorems that establish relations between angles.

A concrete illustration may explain this method:

If CE bisects $\angle C$, and $CD \perp AB$, then

$$\angle DCE = \tfrac{1}{2}(A - B).$$

* Consequently some of the methods for attacking problems (Chapter XV) may be used.

In a beginner's class, let

$$A = m^\circ,\ B = n^\circ.$$

Hence,

$$\angle ACB = (180 - m - n)^\circ.$$

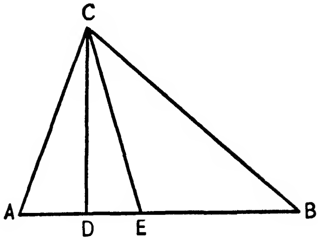
$$\therefore \angle ACE = \left(90 - \frac{m}{2} - \frac{n}{2}\right)^\circ.$$

But

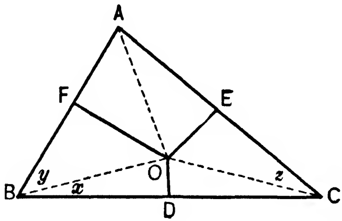
$$\angle ACD = (90 - m)^\circ.$$

$$\therefore \angle DCE = \left(\frac{m}{2} - \frac{n}{2}\right)^\circ$$

$$= \tfrac{1}{2}(A - B).$$



An example which leads to simultaneous equations follows from the theorem: If O is the point of intersection of the perpendicular bisectors of the sides of $\triangle ABC$, then $\angle OBC$ is the complement of angle A.



Let $OBC = x$, $OBA = y$, and $\angle OCA = z$, then we have

$$2x + 2y + 2z = 180^\circ.$$

$$x + y = B.$$

$$y + z = A.$$

$$x + z = C.$$

Solving these equations, we obtain $x = 90^\circ - A$, or x is the complement of angle A.

CHAPTER XII

THE CIRCLE *

REGULAR PROPOSITIONS

Circle or circumference. — Most elementary textbooks, when giving the definitions relating to circle, insist upon a sharp distinction between the terms "circumference" and "circle," denoting by the first the line, by the second the area. While such a distinction is very desirable from the logical point of view, it is not adhered to outside of elementary geometry. In daily life as well as in advanced mathematics, the line is generally denoted by the term "circle," and even the usual elementary textbook soon drops the distinction, and speaks of a circle that passes through three points, or of the intersection of two circles, etc.

While no great harm is done by this differentiation if it is carried out consistently, it is on the other hand distinctly against usage. Since linguistic matters are decided by usage and not by logic, it would possibly be best to denote the line by circle, and to reserve the term "circumference" for the length of this line.

If, however, the elementary distinction is accepted, then it would be logical to define a circumference

* Further discussion of topics relating to the circle may be found in Chapters XIII and XVI.

first, and a circle as an area bounded by a circumference.

First propositions. — The first propositions on the circle, viz., all circles are equal, the diameter bisects the circumference, etc., offer, on account of their simplicity, peculiar difficulties similar to those of the preliminary propositions which were discussed in Chapter V. The remarks that were made there may be repeated here, but it may suffice to mention two points: Do not dwell too long upon these theorems, and reduce their number to the most essential ones.

A proposition, for instance, that is confusing on account of the obviousness of its conclusion and the impossibility of drawing a correct diagram is the following: "A straight line cannot intersect a circumference in more than two points." Many books make this the first proposition, and give a complicated indirect proof, whereas it is far more advantageous to defer it, and to make it a corollary of the theorem, "A circle can be drawn through any three points not in a straight line." From this proposition it follows easily that a circle cannot be drawn through three points in a straight line, and hence that a straight line cannot meet a circle in more than two points.

Several of the fundamental propositions relating to the circle have to be based upon superposition, and the necessity of using this method becomes clear, if we remember that for each new type of figure superposition is the only means for proving equality, as was shown in Chapter VIII. The student will have no

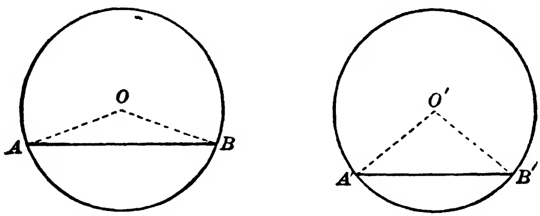
difficulty in finding such proofs if he bears in mind that the parts given as equal must be superposed. Thus to prove that equal central angles intercept equal arcs, we have to superpose the central angles, etc.

Analysis of some theorems. — If the student has mastered the fundamental ideas of analysis as applied in Book I, and if the teacher emphasizes the new typical methods of demonstrating the equality of arcs, chords, etc.,* nearly all demonstrations relating to the circle will be discovered without difficulty. A few examples, however, may be given here. In most cases the students should be able to give not only the answers, but also to propose the questions, and the sequence of the questions.

1. In equal circles the greater chord subtends the greater (minor) arc.

Query. — What is the only means that we know to prove the inequality of arcs ?

Answer. — Unequal central angles.



Query. — What therefore must we prove ?

Answer. — $\angle O > \angle O'$.

Query. — What methods do we know for demonstrating the inequality of angles ?

* See next section.

Answer.—The exterior angle proposition, the proposition relating to one triangle having two sides unequal, and the proposition relating to two triangles.

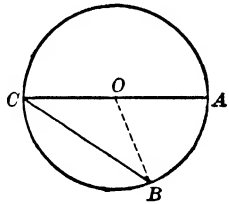
Query.—Which method alone can we use here, and why?

Answer.—The last method, because the angles lie in different triangles.

Query.—Hence what must we show about triangles OAB and $O'A'B'$?

It is easily seen that the triangles satisfy the conditions.

2. To prove that an inscribed angle is measured by one half the intercepted arc, it is better not to propose the general proposition at first, but merely the case in which one side of the angle is a diameter. Usually students find this without help; if not, we may ask as follows:



Query.—Which angle is measured by arc AB?

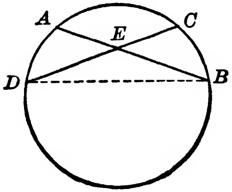
Answer.—$\angle AOB$.

Query.—What relation between angles C and AOB must we therefore prove?

Answer.—$\angle C = \frac{1}{2} \angle AOB$.

The proof is then easily established.

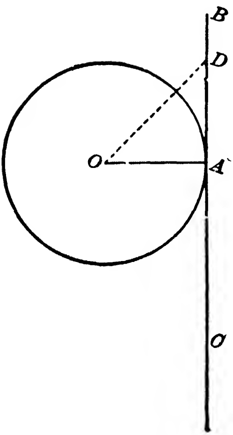
3. If students find it difficult to discover the propositions relating to the measurement of angles by certain arcs, propose at first numerical questions; *e.g.* If arc $AD = 40°$ and arc $CB = 50°$, find $\angle AED$.



After a few such examples, students as a rule will discover the general propositions.

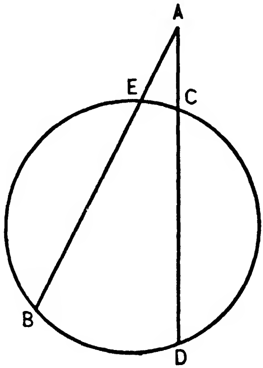
4. In the annexed diagram, which relates to a tangent, students find it sometimes peculiar that we prove that

D lies without the circle. "We can see that *D* is without, why should it be proved?"



Of course, quite in general, "seeing" is not proving, but it can be easily shown in this example that "seeing" will fail to decide the question under certain circumstances. Let the radius be very great, say, several miles, and *AD* very small, say, one inch. Then nobody could decide by seeing whether or not *D* is without, while the demonstration shows it beyond a doubt.*

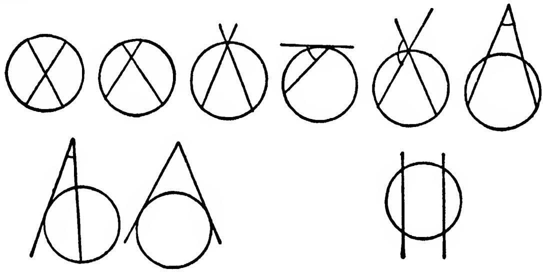
Generalizations of certain theorems. — The theorems relating to the measurement of angles by arcs can be generalized as follows. Let *A* be such an angle, and let it be generated by a counter-clockwise rotation of *AB*, from the initial position *AB* to the terminal position *AD*; then the point of intersection of moving line and circle would sweep over arc *BD* counter-clockwise, and over arc *EC* clockwise. If we consider all arcs as positive if the moving point travels over them counter-clockwise, and as



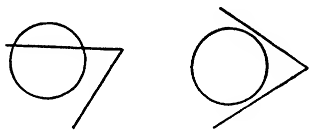
* The theorem, The central angle is measured by the intercepted arc, will be analyzed in the chapter on limits (XIII).

negative if the moving point travels over them clockwise,* we may summarize all theorems relating to angles as follows: An angle is measured by one half the algebraic sum of the intercepted arcs.

Thus the notion of positive and negative geometric quantities, that is so widely used in modern geometry, enables us to merge into one proposition a number of different theorems which are illustrated in the annexed diagrams.



The proposition indicated by the fifth diagram is generally not given in textbooks; the last diagram indicates that if the angle is zero, *i.e.* if the lines are parallel, the algebraic sum of the arcs is zero.



If we widen our definition by admitting imaginary arcs, the proposition is true even if one or both sides of the angle do not meet the circumference at all. Thus if the vertex of an angle moves over the entire plane and its sides rotate in any manner, the proposition

* For a somewhat simpler, but less logical, distinction between positive and negative arcs, see Schultze and Sevenoak's Geometry.

always remains true. It does not change abruptly at any particular point, but is continuous all over the plane. The principle implied here is often referred to as the principle of continuity.

EXERCISES

The general methods of Book II should be practiced until they become quite familiar to the students. Some of the most important are:

1. The equality of arcs is usually demonstrated by means of

   (*a*) equal central angles,
   (*b*) equal chords.

2. The equality of chords is usually demonstrated by means of

   (*a*) equal arcs,
   (*b*) equal distances from the center.

3. The inequality of arcs and chords is proved in an analogous way.

The teacher should take up each of such methods separately and illustrate it by many theorems, and sometimes by problems. This matter, however, is too simple to require further detail.

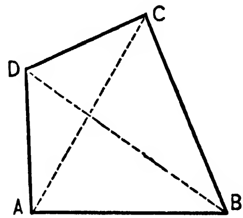
The theorems on the measurement of angles by arcs should be illustrated by many numerical exercises. Simple exercises of this sort are easily constructed by considering certain figures, as, for instance, an inscribed quadrilateral, a circumscribed triangle, an inscribed pentagon, and assigning numerical values to certain

angles and arcs, and requiring the numerical values of other angles or arcs.

For certain difficult exercises of Book II it is sometimes necessary to prove that four points are concyclic, *i.e.* that a circumference can be drawn through them. Two theorems may be used for this purpose. The vertices of quadrilateral $ABCD$ are concyclic if

(*a*) $\angle ADB = \angle ACB$, or

(*b*) $\angle A$ is the supplement of $\angle C$.



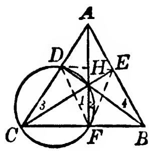
Each theorem may be proved by the indirect method.

Some originals that may be easily proved by means of concyclic points, which are otherwise exceedingly difficult, are the following:

1. If in quadrilatral $ABCD$ (see above diagram) $\angle ADB = \angle ACB$, then $\angle DCA = \angle DBA$.

2. In the same diagram, if

$$\angle B + \angle D = 180°, \text{ then } \angle BAC = \angle BDC.$$



3. Each angle formed by joining the feet of the three altitudes of a triangle is bisected by the corresponding altitude.

4. If from any point in the circumference of a circle perpendiculars be dropped upon the sides of an inscribed triangle (produced if necessary), the feet of the perpendiculars are in a straight line.

CHAPTER XIII

LIMITS

DOGMATIC TREATMENT OF LIMITS

No part of elementary geometry has aroused such an extensive and heated controversy as the theory of limits. It is the favorite topic of those who lay all the poor results of geometric teaching to lack of rigor, and who until recently controlled the situation to such an extent that hardly anybody dared to express an opposite opinion.

The theorems that are involved in this controversy belong to two groups:

1. Those containing the so-called incommensurable case.

2. The determination of the circumference and the area of a circle. (In solid geometry the surface and volume of cone, cylinder, and sphere.)

The first group involves the idea of incommensurable number, which was put on a scientific basis only in the nineteenth century. The other involves the measurement of a curved line, a problem for which our usual method of measuring (*i.e.* laying off the unit of length) utterly fails, and which leads to other, almost metaphysical, difficulties.

There exists no rigorous treatment of these matters that is suitable for secondary schools, and the belief

that the treatment given in "rigorous" school text-books is really rigorous is a delusion. The influence of the dogmatists, however, and their argument that only ignorance prevented teachers from using rigorous methods, were powerful enough to make teachers and authors vie with each other in making the treatment of limits more and more complex.

Often highly artificial definitions were given, which conveyed no meaning to the students, to be followed by eight or ten abstract theorems relating to the product, the quotient, etc., of variables, and finally a complex, non-conclusive proof for the equality of the limits of equal variables.

In this rigorous fashion the subject was treated throughout the entire course. True enough, certain students could repeat the words, but not one in a hundred had a clear notion of what he was really doing. The absolute inefficiency of this way of teaching this subject is attested by nearly all college teachers who have occasion to make use of limits in advanced courses.

Fortunately the pendulum is commencing to swing the other way, and the conviction is gaining ground more and more that it is a mistake to treat this subject in the way in which it has been treated.

RATIONAL TREATMENT OF LIMITS

The incommensurable case. — There can be no doubt that under the conditions that prevail to-day in our schools, it would be better to omit all proofs relating to

limits, than to attempt to give the customary "rigorous" demonstrations.

It is possible, however, to give to the student a fair understanding of the nature of the problem and its difficulties by treating the subject very concretely. To illustrate by a definite example, let us consider the first proposition, which under the customary arrangement requires limits, viz.: "A central angle is measured by the intercepted arc." Usually this proposition is given as a corollary of the theorem establishing the proportionality of central angles and intercepted arcs. But as the main theorem is never applied, we may omit it, and at once attack the corollary in the following manner:

Since equal central angles intercept equal arcs, each central angle of $1°$ intercepts $\frac{1}{360}$ of the circumference. Hence, we make a circumference equal to $360°$ and we obtain:

An angle of $1°$ intercepts an arc of $1°$.
$\therefore$ an angle of $(\frac{1}{20})°$ intercepts an arc of $(\frac{1}{20})°$.
$\therefore$ an angle of $(\frac{37}{20})°$ intercepts an arc of $(\frac{37}{20})°$.

Or more generally,

An angle of $\left(\frac{1}{m}\right)°$ intercepts an arc of $\left(\frac{1}{m}\right)°$.

$\therefore$ an angle of $\left(\frac{n}{m}\right)°$ intercepts an arc of $\left(\frac{n}{m}\right)°$.

Therefore all central angles that can be expressed as common fractions — proper or improper — are measured by their intercepted arcs.

The question whether this demonstration was quite general, *i.e.* if *any* central was measured by the inter-

cepted arc, would very likely be answered by the student in the affirmative. This would lead naturally to the subject of incommensurable number. The student easily sees that certain numbers, *e.g.* $\sqrt{2}$, cannot be fractions, for if $\sqrt{2}$ were equal to $\frac{m}{n}$ (where $\frac{m}{n}$ is in its lowest terms), then $2 = \frac{m \cdot m}{n \cdot n}$, which is impossible since m and n have no common factor.

It could be pointed out then that there are other numbers that cannot equal common fractions, as $\sqrt{7}$, $\sqrt[3]{5}$, $3.14159 \cdots$ or π, and the term "incommensurable number" could be introduced.

If the question whether our geometric theorem of the central angle was proved for all cases, were *now* repeated, we would quite likely receive a negative reply. We may then either tell the students that the theorem can be proved for incommensurable numbers also, but that this proof is too difficult for school work; or, we may attempt to make the incommensurable case more plausible by considering approximations of one of these numbers, for instance, the following approximations of $\sqrt{2} = 1.4, 1.41, 1.414, 1.4142$, etc. Obviously the theorem is true for all approximations, hence the two numbers — the numerical measure of the angle, and the numerical measure of the arc — cannot differ respectively by .1, .01, .001, .0001, etc. Or the error cannot be as large as any number, however small, which we may assign.

We have thus proved that there can be no *finite* dif-

ference between the numerical measurements of angle and arc, and this is really all that the so-called rigorous proofs with their complete machinery accomplish.

It we have a craving for rigor, and consider the above method less exact than the customary way, we could make it fully as rigorous as any one of the usual books, by introducing the following *definition* of equality of incommensurable numbers: Two incommensurable numbers are equal if all their approximations are equal. This would practically do away with the incommensurable cases.

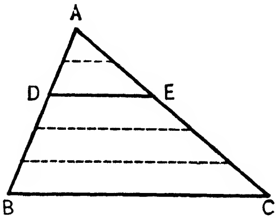
The essential character of the above method is the fact that it approaches the idea of incommensurable from the purely arithmetic side, and that it dispenses with limit, commensurable and incommensurable lines, etc. This is not as truly geometric as the customary mode, but since we placed the entire theory of ratio and proportion upon an arithmetical basis,* there can be no sound reason against doing the same for these special and difficult cases of ratio. The purely geometric idea of incommensurable is so difficult to the student because he never before treated these matters geometrically. He has not found geometrically the common measure of two lines, he has no *geometric* evidence that there are lines that have no common measure, etc.

To illustrate the use of this arithmetical plan further let us consider the theorem, A line parallel to one side of a triangle divides the other two proportionally.

Suppose in the annexed diagram that $AD = \frac{2}{5} AB$, *i.e.*

* See Chapter XIV, Ratio.

if AB is divided into five equal parts, then AD equals two of these parts. If through the points of division the parallels to BC are drawn, then obviously AE is $\frac{2}{5}$ of AC.



Or if $$\frac{AD}{AB} = \frac{2}{5},$$

then $$\frac{AE}{AC} = \frac{2}{5}.$$

Therefore the theorem is proved if AD can be expressed in the form $\frac{n}{m} AB$, when $\frac{n}{m}$ is a common fraction.

And again we point out that only rational numbers can appear in the form $\frac{n}{m}$, and treat the incommensurable case as before.

Of course the commensurable case thus treated is slightly more complex than in the customary way, but it does away with the bugbear "common measure." For the incommensurable case, however, this method is undoubtedly preferable, since it does away with commensurable and incommensurable lines, variables, limits, etc.*

Length of a curve. — While the theorems considered in the preceding section can very well be taught with-

* If external conditions compel the teacher to use the customary mode of proving the incommensurable case, he may first teach one or two propositions in the arithmetical manner and then lead over to the "geometric-rigorous" way. He may introduce commensurable and incommensurable lines as lines whose ratios are commensurable or incommensurable numbers, derive therefrom the fact that for two commensurable lines there always exists a third line, which is contained in each without remainder, etc.

out any reference to "limit" and similar terms, the length of curves cannot dispense with the notion of limit. Here, however, lies the solution of the pedagogic difficulty in a secondary school in the abandoning of the demonstration.

Thus, it would be better that the proposition, The circumference is the limit of an inscribed polygon, etc., were not proved at all, for to the mind not used to mathematical subtleties this appears to be axiomatic. Indeed, it requires a good deal of training to recognize that we have to deal here with a difficult proposition which needs a proof. Not one student in a hundred will acquire a clearer understanding of this theorem by studying the proof, or rather the series of proofs that is required for this proposition. The fact that a circle is the limit of a polygon, etc., appears to be so plain that it forms one of the best illustrations of the notion of limit, at the time when this definition is first studied.

It is an old experience that many problems appear as problems only after considerable advance in their study has been made. To the unsophisticated savage objects are "heavy"; this is a fundamental fact not requiring explanation, and only after considerable progress has been made do we recognize that we have to deal here with the problem of gravitation that requires solution. To the layman there is no difficulty in explaining why some particular house appears to be red. The house *is red*, he will tell us, and only considerable reflection will show him that he has to deal with the mystery of sensation. In all such cases, we must first teach the

student that *there is a problem*, and only after this has been accomplished may we try to give a solution of the problem. It is precisely the same thing with the length of curved lines. The young student in imagining the length of a curve thinks of the length of a curved string that can be straightened. There is nothing difficult to him about this problem, and only considerable study will show to him that there is a problem, and a difficult one.

In solid geometry, it is the usual practice to assume that surfaces are limits — although here too some dogmatists have tried to display their learning by objecting. Why not use it for plane geometry, also? It is, however, advisable in all such cases to explain to the students that this plan involves a distinct assumption.

General suggestions. — If teachers are compelled to teach limits and all general matters connected with this notion, the following suggestions may be helpful:

1. Give the definition of limit in an algebraic form, making the variable equal to x, the constant equal to some other letter.

2. Be satisfied with an approximate definition. The student is not familiar with the idea of a function and two interdependent variables. It is difficult enough for him to imagine one variable. Do not reject a definition because it does not cover all the cases which the student will meet in his college work. College teachers may be inconvenienced by this, but it is far more important for them to get students who think well, than those who *know* exact definitions. It is a very common experience in mathematics, that certain defini-

tions must be given at first in a preliminary form. Give the definition which is best for your own purpose, *i.e.* the teaching of elementary geometry, and let college teachers revise these later on. If, for instance, a teacher should feel that the clause, "A variable cannot reach its limit," is helpful to his work, he should accept it. All limits considered in elementary geometry have this peculiarity, and it may be advantageous to point it out in the beginning. Certainly the fact that there are other limits, outside of school geometry, that have not this peculiarity, should have no weight in the decision.

3. Illustrate the nature of a limit by a large number of concrete cases. Recurring decimals, as $.999 \cdots$; infinite series, as $1 + \frac{1}{2} + \frac{1}{4} \cdots$; algebraic functions, as $\left(\frac{1}{2^n}\right)_{n \doteq \infty}$; etc., offer good examples.

Also motion examples, the fact that circumference and area of a circle are respectively limits of perimeter and area of certain polygons may be used.

4. Do not attempt to teach in the beginning the general theorems relating to products of variables, quotients of constants and variables, quotients of variables, etc.

5. Do not teach the geometric proof that is usually given for the theorem, "If two variables that have limits are always equal, their limits are equal." The proof is not conclusive and is rarely understood by students.

6. Do not dwell too long upon these topics. The longer you do, the more confused will students become.

CHAPTER XIV

THE THIRD BOOK OF GEOMETRY

THEOREMS RELATING TO PROPORTIONS

Ratio and proportion. — The modern school book defines a ratio as a fraction. The finding of a ratio and the determination of a quotient are identical problems. This, however, in many examples involves the notion of irrational numbers, and we can understand that, at a time when only rational numbers were recognized and irrational numbers were considered as impossible, the definition of a ratio as a quotient was considered incomplete. Thus Euclid and the other ancient geometers did not use the arithmetical definition of a ratio. Even to-day some authors reject it, because it is non-geometric, it "constitutes a break in the logic of the geometry," it leads to multiplying lines, etc.

But even admitting that the arithmetical definition is less scientific, its pedagogic advantages are so great that there can be no question as to whether or not it should be used in our secondary schools. One has only to read Euclid's definition * to be convinced that it is utterly unfit for young students.

* "The first of four magnitudes is said to have the same ratio to the second, which the third has to the fourth, when any equimultiples whatsoever of the first and third being taken, and any equimultiples whatsoever of the second and fourth; if the multiple of the first be less than that

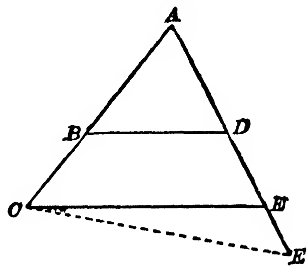
The fact that the arithmetical view of ratio leads to a product of two lines is no serious objection, since all difficulties may be removed by defining the product of two lines as the product of their numerical measures. It would lead us too far to discuss the teaching of ratio and proportion in detail; here only one remark may find place. The students should understand that a ratio involves a comparison of two quantities, *e.g.* the statement $a : b = 7 : 1$ means a is 7 times b. As students are inclined to use ratio and proportion in an utterly mechanical fashion, it happens sometimes that they forget what a proportion really is. Frequent numerical illustrations will diminish this difficulty. Thus after we prove that $AB : BC = AD : DE$, we should ask, If $AB = 3\ (BC)$, what is the relation between AD and DE? This should of course not be restricted to a case as simple as the one presented here, but every proposition relating to proportion should be illustrated numerically.

Remarks on certain theorems. — 1. The proposition relating to the segments made by a line parallel to one side of a triangle is of fundamental importance. Its proof, like any proof that establishes proportionality of a new type of figure, necessarily leads to the incommensurable case,* although this may be avoided if proportional areas are studied before.

of the second, the multiple of the third is also less than that of the fourth: or, if the multiple of the first be equal to that of the second, the multiple of the third is also equal to that of the fourth; or, if the multiple of the first be greater than that of the second, the multiple of the third is also greater than that of the fourth."

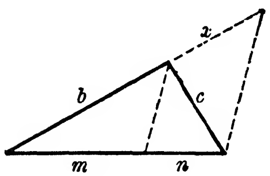
* For discussion of this proof see Chapter XIII.

It is worth noticing that this theorem is true whether the parallel meets the sides or their prolongations. In the former case we have internal, in the latter external, division.



2. In demonstrating the converse of the preceding proposition, usually a line parallel to CE is drawn through B. The proof, however, becomes simpler if a parallel to BD is drawn through C (see annexed diagram).

3. In analyzing the proposition of the bisector of an angle of a triangle, *i.e.* $m : n = b : c$, it appears difficult to make students discover the construction of the line parallel to the bisector. The matter, however, becomes simple, if we make it a problem by requiring the construction of the fourth proportional to m, n, and b. This produces the construction of the necessary lines, and it remains only to demonstrate the equality of x and c, which is not at all difficult.



4. The corresponding proposition of the bisector of the exterior angle is of no great importance unless harmonic division is studied. If it is taught, however, students can be led to the discovery of the proof by the absolute analogy of this proposition with the preceding one. With proper lettering one proof may be written that fits both theorems.

5. For proving some theorems on similarity of triangles, a new method for establishing the equality of lines is frequently applied, viz., the two lines are made corresponding terms of two proportions whose other terms are respectively equal. Thus if

$$a : b = x : c,$$

and $$a : b = y : c,$$

then $$x = y.$$

If this method is emphasized and practiced, some theorems lose a great deal of their difficulty, *e.g.* the proposition, Two triangles are similar if their homologous sides are proportional.

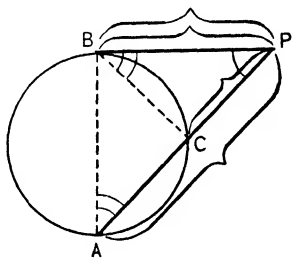
6. In regard to the propositions relating to segments which we obtain if we draw two intersecting chords, or two bisecting secants, or a secant and a tangent, it may be well to point out that the second and third propositions are only special cases of the first. The secants may be considered as chords that are divided externally, and the tangent represents that limiting case of a secant, for which secant and external segment become identical.

EXERCISES RELATING TO PROPORTIONAL LINES

Simple exercises. — Similar triangles are of the same fundamental importance in proving the proportionality of lines, as equal triangles are in proving the equality of lines. Hence, we may say, **The proportionality of lines is in most cases proved by means of similar triangles.** Since the greater part of Book III refers to proportional lines, and since most numerical calculations of

the lengths of lines depend upon proportional lines, the matter is of great importance. The details of the mode of procedure may be described as follows:

1. *Let the student mark the four lines forming the proportion.* A brace is a very practical symbol for pointing out each line, and two such braces should be drawn if a line occurs twice in the proportion. Thus to prove $PA : PB = PB : PC$ we should use the annexed marks.



2. *Select two triangles so that each contains two of the given lines.* In a few cases, this can be accomplished in several ways, and it may happen that the student selects the wrong pair of triangles. In such a case the impossibility of demonstrating the similarity of the triangles should cause the student to select another pair.* Such cases, however, are very rare. In the above diagram, the triangles are obviously PAB and PBC.

3. *Prove the similarity of the two triangles.* In most cases this is accomplished by means of equal angles. The equal angles should be marked as indicated in the above diagram.

4. Write the proportion, choosing as first term the

* The maximum number of triangle pairs which can be formed from the lines a, b, c, and d is three, viz. (a, b) and (c, d); (a, c) and (b, d); (a, d) and (b, c). Of these three pairs two can be used for the proof, and the third pair cannot be used.

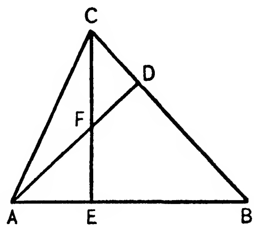
first term of the conclusion. Even then the sequence of terms in the resulting proposition may not be identical with that of the conclusion, but alternations will easily remedy this defect. To obtain the homologous sides, we should pay attention to the marks of the angles. Thus, in the above diagram PA is included by the two marked angles of the large triangle, hence PB, which lies analogously in the small triangle, is the homologous side. PB in the large triangle, and PC in the small one, are homologous because they are opposite the angles marked by two marks.

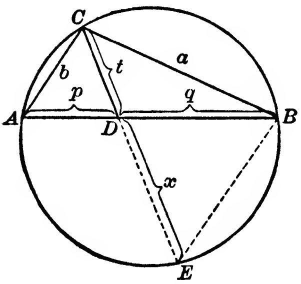
5. *If we have to prove that the product of two lines equals the product of two other lines*, the mode of procedure is precisely as above, only we have to equate the product of the means and the product of the extremes of the resulting proportion.

Construction of originals. — A great many exercises of this type should be solved until every student is fully acquainted with this method. If textbooks do not give a sufficient number of these originals, it is an easy matter to construct them.

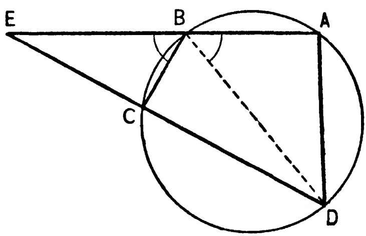
First, a great many regular book proportions that follow later are theorems of this type, and it is expedient to solve nearly all of them to illustrate our method. To avoid the student's referring to the text, it is possibly best to use them as illustrations for oral work when the method is first explained. Such propositions are those of intersecting chords, of intersecting secants, of a tangent and a chord, of the right triangle, of the product of altitude and diameter of circumcircle.

Secondly, the study of a type figure and the determination of all possible proportions in such figure produces a great many exercises. For instance, if, in the annexed diagram, the altitudes *AD* and *CE* meet in *F*, we have four similar triangles in the diagram, viz., *AFE*, *ABD*, *CBE*, and *CFD*. Taking all possible combinations, we obtain 6 pairs of similar triangles ($1 \sim 2$, $1 \sim 3$, $1 \sim 4$, $2 \sim 3$, $2 \sim 4$, and $3 \sim 4$), and since each pair of triangles produces 3 proportions, we have 18 proportions in this diagram, and of course 18 products of lines equal to 18 other products of lines.



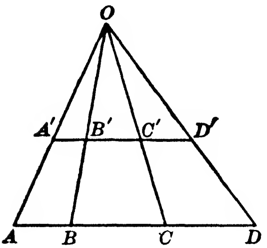


Similarly, if in the annexed diagram $\angle ACD = \angle ECB$, we obtain 3 similar triangles (*ACD*, *EBD*, and *ECB*), and consequently 9 proportions. In the annexed diagram, if $\angle EBC = \angle ABD$, then three triangles are similar (*EBC*, *EDA*, and *DBA*), and there are consequently nine proportions.

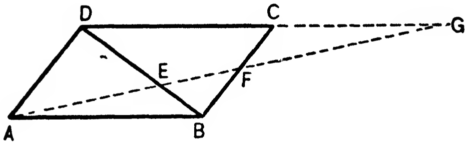


The altitude drawn upon the hypotenuse of a right triangle produces a figure in which nine proportions may be proved, etc.

Difficult exercises. — In the more difficult exercises relating to the proportionality of four lines, it is impossible to find two similar triangles so that each contains two of the given lines. In such cases we have to find a third ratio which may be proved to be equal to each of the given ratios. In other words, these theorems are resolved into two of the preceding kind. Thus, if $AB \parallel A'B'$, and it has to be proved that $\frac{AB}{A'B'}=\frac{BC}{B'C'}$, our regular method obviously fails, since there exist no triangles which contain two of these lines. But it is easy to prove that each of the given ratios equal $\frac{OB}{OB'}$.



Or if $ABCD$ is a parallelogram and we have to prove $\frac{EF}{EA}=\frac{EA}{EG}$, we have again the same difficulty. But $\frac{BE}{ED}$



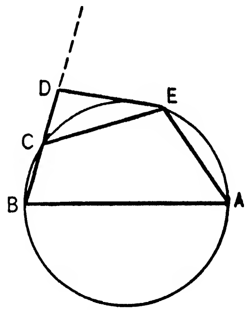
can easily be proved, by means of the regular method, to be equal to each of the given two ratios.

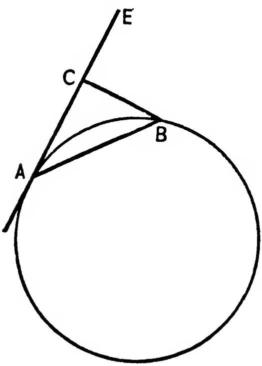
Numerical examples.— Each regular proposition relating to proportionality of lines should be illustrated by numerical examples. By such work we familiarize the student with the facts stated in each proposition, we make sure that he understands the meaning of such

propositions, and we obtain an abundance of simple and concrete exercise material, that frequently can be used for applied problems (see Chapter XVIII).

Each of the originals on proportional lines may be accompanied by a numerical question to show the usefulness of proportions. Thus, after you prove that the diagonals of a trapezoid divide each other proportionally, assign numerical values to three of the segments and require the fourth, or assign values to the second and third terms, and let the fourth be twice the first, and require the first.

Exercises of this type are not difficult, if the proportion upon which they are based is given; they become somewhat harder if they have to be solved without such proportion. *E.g.* two sides of a triangle are ten and twelve, the altitude upon ten equals eight, find the altitude upon twelve. Here the student has to discover the proportion.





Still more difficult are the exercises, if one or both of the necessary triangles have to be constructed. *E.g.* if $DC = 3$, $EC = 5$, $EA = 6$, $ED \perp DB$, and AB is a diameter, find AB. Or still more difficult: If AC is a tangent, $BC \perp AE$, $AC = 4$, $AB = 6$, find the diameter of the circle.

METRICAL RELATIONS BETWEEN LINES

Propositions establishing metrical relations between lines. — Before we studied proportions we could establish only the equality or inequality of lines. The introduction of proportions makes possible the calculation of the numerical values of lines. As this leads to a number of applied examples, such as the finding of heights and distances, the subject is not without interest.

Theorems of this type are all those that establish an equation involving the lengths of lines, in particular the Pythagorean theorem, the median proposition, the angle-bisector theorem, etc., etc. To the ancients all such propositions meant relations between the areas of rectangles and squares, while to us they represent algebraic equations between the numerical measures of lines. Thus $t^2 = ab - pq$ meant to Euclid that a square is equal to the difference of two rectangles, while modern writers are not afraid to apply algebra to geometry, and to consider the above statement as a simple algebraic equation involving five numbers, t, a, b, q, and g.

The Pythagorean theorem. — The best-known example of this type is the theorem which connects the lengths of the three sides of a right triangle ($a^2 + b^2 = c^2$) that was proved by Pythagoras about 550 B.C. This proposition is probably the most widely applied theorem of the entire geometry; in fact, it seems to be the only theorem that is frequently applied. Its great importance has always been recognized, and it is not surprising that many people have attempted to discover new proofs of

this proposition, so that at present more than 100 are known.

For the beginner the algebraic treatment is possibly the best. Prove that each arm is a mean proportional, and derive therefrom the value of the square of each arm, etc.

Euclid's proof, which is of course purely geometrical, depends upon the equality of two triangles and is — chiefly on account of its historic interest — still given in many textbooks. Very interesting are the proofs which cut the squares on the arms into parts which properly united produce the square on the hypotenuse.

A proof which deserves mentioning on account of its brevity, although it does not fit into the arrangement of most textbooks, is the following one:

If $\triangle ABC \simeq Kc^2$,

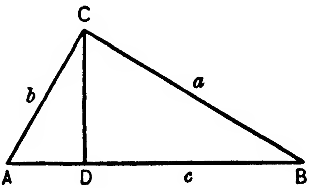
then $\triangle ADC \simeq Kb^2$,

and $\triangle BCD \simeq Ka^2$.

Hence obviously,

$$Ka^2 + Kb^2 = Kc^2,$$

or $$a^2 + b^2 = c^2.$$



Analysis of theorems of this type. — The inductive sequence makes it a very simple matter to discover theorems of this type and their proofs. To discover, for instance, the theorem of the square of a side of a triangle opposite an acute angle, let us form a series of numerical and algebraic exercises, starting with applications of the Pythagorean theorem and ending with the required theorem. This may be done as follows:

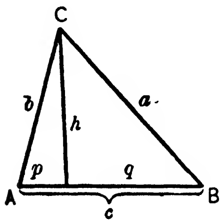
1. Give numerical or literal values to two sides of a right triangle and require the value of the third side.

2. Give numerical or literal values to the base and an arm of an isosceles triangle, and require the altitude.

3. Assume altitude and one side of an isosceles triangle and require the remaining side.

4. Find the altitude of an equilateral triangle whose side is given.

5. Find the side of an equilateral triangle whose altitude is given.



6. If in the annexed diagram $h \perp c$, $b = 10$, $h = 8$, and $a = 17$, find c.

7. In the same diagram, if $b = 10$, $h = 8$, $c = 14$, find a.

8. In the same diagram, express a in terms of b, h, and c.

9. In the same diagram, if $a = 20$, $b = 37$, $q = 16$, find p.

10. In the same diagram, express p in terms of a, b, and q.

11. In the same diagram, if $b = 15$, $p = 9$, and $c = 25$, find a.

12. In the same diagram, express a in terms of b, c, and p.

Thus the student arrives at the required proposition, and has in addition acquired some facility in attacking such propositions in general, — a matter far more important than the knowledge of a single demonstration.

Work of this kind is greatly facilitated by the use of simple notation. We should, as far as possible, designate lines by a single letter, denoting the three sides of a triangle by a, b, c; the altitudes by h_a, h_b, h_c; the medians by m_a, m_b, m_c. For the notation of triangles, if quite a number occur in a proof, do not use two letters, but Roman numbers, as I, II, III, etc.

Use of directed lines. — The proposition of the square of the side of a triangle furnishes another example,

showing how, by the use of directed lines, we may, by one statement, cover several cases which otherwise would appear as a number of different propositions.

If we consider the projection p of the side b upon the side c as positive if it lies in the same direction as c, otherwise as negative, then the square of any side a of a triangle may be expressed by the formula:

$$a^2 = b^2 + c^2 - 2\,cp.$$

If the angle opposite a is 90°, then the third term of the formula vanishes; if the angle opposite a is obtuse, then p becomes negative, and the third term positive.

The advantage of such a general formula is especially obvious in all examples in which the character of the angle is not known. Thus, the finding of p when the three sides are given would without the general formula require a special investigation into the character of a certain angle, before we could decide which formula may be applied. The general formula, however, may always be applied, and the resulting value of p will inform us whether the angle opposite a is acute, right, or obtuse.

Use of formulæ. — Students should be thoroughly familiarized with the fact that a formula may be used, not only to find the quantity which it expresses explicitly, but also for the purpose of finding any one of the quantities involved when the others are known. Thus the formula:

$$a^2 = b^2 + c^2 - 2\,cp \qquad (1)$$

should be used, not only for the finding of a, but also for finding b and c, and in particular for the finding of p when a, b, and c are given. Some teachers derive a corollary expressed by the formula

$$p = \frac{b^2 + c^2 - a^2}{2c}, \tag{2}$$

then let students memorize (2), and solve all examples by means of this formula. But such a course can hardly be recommended. It is practically just as simple to find a numerical value of p by means of (1) as by means of (2). Hence there is absolutely no reason why the student's memory should be burdened with an unnecessary formula. But, furthermore, by such practice the student is led to believe that the application of a formula is restricted to the finding of the quantity which it expresses explicitly, a belief that may form quite a hindrance in more advanced work. That such a belief is not uncommon among students, we may observe quite frequently. Propose, for instance, to students in solid geometry who have just derived the formula for the altitude of a regular tetraedron H in terms of the edge a, $\left(H = \frac{a}{3}\sqrt{6}\right)$, the opposite problem, viz., to find the edge in terms of the altitude, and a great many students will start an independent geometric investigation instead of using the formula just found. The defect of such a method is particularly obvious when the opposite problem is very difficult, as the finding of the edge of a regular tetraedron whose volume equals V.

The student should become aware that all such problems can be solved algebraically, if we succeed in establishing any algebraic relation between the two quantities involved, and that hence the two problems, viz., the finding of the edge if the volume is given, and the finding of the volume if the edge is given, are *geometrically* alike.

Unnecessary corollaries. — The reasons advanced in the preceding paragraph apply with equal force to all succeeding propositions, such as the median theorem, the proposition of the bisector of an angle of a triangle, etc. The practice of deriving and memorizing corollaries that give explicit formulæ for median, angle-bisector, etc., cannot be recommended. In all such cases the student should memorize only one formula which expresses the fundamental theorem, and solve all numerical examples by substituting the given values in this formula. Thus, for the median proposition and all its applications we need only one formula, viz.:

$$2a^2 + 2b^2 = 4m_c^2 + c^2,$$

and similarly for other theorems.

An exception to this rule, however, is the altitude formula:

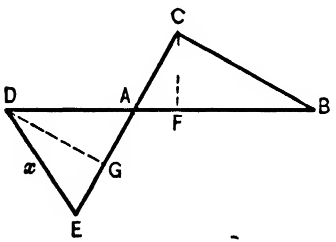
$$h_c = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}.$$

Here, also, we may, in a concrete numerical example, find the numerical value of the projection p and then of h, but the numerical difficulties of such a procedure are often considerable, while the application of the formula is quite simple. Moreover, the altitude is so often

needed in more advanced work that the memorizing of the formula seems to be justified.

Projections. — Numerical examples based upon the above propositions derive a certain interest from the fact that they may be applied to practical examples. A great many of these make use of the "projections," but it seems that the great usefulness of this concept is not generally recognized by teachers. It is sometimes believed that certain problems require trigonometry for their solution, while they can be easily solved by pure geometry, if only the proper use of projections is made.

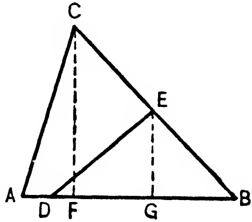
If, for instance, in the annexed diagram two lines DB and EC meet in A, and the values of AB, BC, CA, AE, and AD are given, and DE is required, then trigonometry would give us $\angle BAC$, which equals DAE, and again trigonometry would give us DE or x from $\triangle ADE$. In geometry we do not need to find the angle itself, but the projection of a segment of one side of the angle upon the other side. Thus in the above diagram we find AF, the projection of AC upon AB. Similar right triangles give us the corresponding projection AG in $\triangle ADE$, and hence we easily find DE or x.



Whenever an angle is found by a trigonometrical solution as an intermediate result only, to be used again for the calculation of lines, the same work can usually be accomplished with the aid of pure geometry by in-

troducing projections. Such a projection takes the place of the angle (or rather its trigonometric function).

To illustrate further: If the three sides of a triangle ABC, and the distances BE and BD are given, and DE is required, we would proceed trigonometrically by first solving $\triangle ABC$, obtaining thus $\angle B$, and then obtain DE by solving $\triangle DBE$. Angle B would be an intermediate result only. In plane geometry we introduce the projection FB instead of (the trigonometric function of) angle B, find by similar right triangles the corresponding projection GB, and finally obtain DE from $\triangle DEB$.



This method may be used for some of the regular propositions, *e.g.* the median proposition. To find m_c in terms of a, b, and c, determine first p in the projection of b upon c, and then from $\triangle\left(b, m, \frac{c}{2}\right)$ determine m.

Obviously we could find by this method the length of a line m that does not bisect c, but that divides it in any given ratio. If m divides c in the ratio $k : l$, then it is not difficult to prove that

$$m^2 = \frac{ka^2 + lb^2}{k + l} - \frac{klc^2}{(k + l)^2}.$$

Relation between angle and projection. — In most cases the value of a line and its projection will not lead to the numerical value of the included angle; and *vice versa*, the value of the angle will not enable us to calculate the

projection. The few exceptional cases, however, should be known to the teacher. For the secondary school student, it is sufficient to know how to find the projection of one line upon another if the included angle is 30°, 45°, or 60°, and their respective supplements. It is, however, possible to calculate the projections for all angles which equal the central angles of regular polygons that can be constructed with rules and compasses, *e.g.* 15°, 24°, 18°, 22°, 30°, etc.* Angles the number of whose degrees is integral make such calculations possible, if they are multiples of 3°, as 6°, 9°, 12°, etc.

Among the easiest ones to derive is the projection p of a line a, if p and a include an angle of 22° 30′, viz.,

$$p = \frac{a}{2}\sqrt{2+\sqrt{2}}.$$

Construction of exercises. — To construct exercises of this type we may assign numerical values at random to certain lines, *e.g.* the sides of a triangle, and require the numerical value of other lines, as the median angle-bisector, etc. The only difficulty that may arise is the complexity of the numerical work, and for obtaining simpler, especially rational, results the following formulæ may be useful.

Right triangle.—If m and n are positive integers, which are relatively prime, if $m > n$, then rational values of three sides of a right triangle may be obtained by the formulæ: †

* See Chapter XVI.

† If we wish to obtain values that have no common factor, m and n should not both be odd.

$$a = m^2 - n^2.$$
$$b = 2\,mn.$$
$$c = m^2 + n^2.$$

Thus, the values 3, 4, 5; 5, 12, 13; 15, 8, 17; 7, 24, 25; 21, 20, 29, etc., may be obtained.

The values 5, 4, and 3 are well known and may be used to construct a right angle. If we have a right triangle the ratio of two of whose sides conforms with these numbers, it is easy to find the third. Thus, if $c = 55$, $a = 44$, then without further calculation it follows that $b = 33$.

Triangle containing an angle of 60°.—If in $\triangle ABC$, $\angle A = 60°$ and a lies opposite A, then the following formulæ produce rational sides:

$$b = m^2 - n^2.$$
$$c = (2\,m - n)n.$$
$$a = m^2 - mn + n^2.$$

Thus, we find 5, 8, 7; 3, 8, 7; 8, 15, 13; 9, 24, 21; 11, 35, 31, etc.

If $A = 120°$, we change b to $m(2\,n - m)$, but leave a and c unaltered.

Median.—If m, n, p, and q are positive integers, and a, b, and c the sides of a triangle, then the following formulæ produce triangles with a rational m_c:

$$x = mp + (2\,m + n)q.$$
$$y = (m + n)p + nq.$$
$$z = np + 2(m + n)q.$$

Similarly formulæ may be given for triangles that

have two or three rational medians, rational altitudes, bisectors, or areas, or pyramids having rational volumes, etc.

Readers wishing information on this subject may consult the chapter on indeterminate equations in the textbooks on higher algebra.

CHAPTER XV

METHODS OF ATTACKING PROBLEMS

LOCI

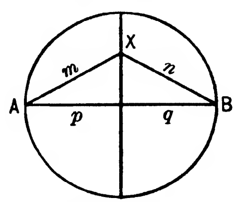
Scientific aspect of loci. — *Definition.* — The two conditions that must be satisfied to make a line the locus of a point are usually stated as follows:

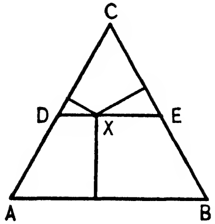
1. Every point in the line must satisfy the given condition.

2. No point without the line must satisfy this condition.

The second part is not a pedantic, unnecessary clause without a practical value, but an absolutely essential part of the definition. If we omitted it, we would in some cases be led to consider a part of a locus as a complete locus, while in others we would obtain a locus when there is none. The following two examples illustrate these points:

1. If we denote the distances of a point X from two fixed points A and B by m and n respectively, and the projections of m and n upon AB by p and q, then every point in the perpendicular bisector of AB satisfies the condition $m^2 : n^2 = p : q$. Hence, if we did not examine points without we would consider the perpendicular bisector the locus of the point X, which satisfies the above condition ($m^2 : n^2 = p : q$), and we would not obtain the other and more interesting part of the locus, viz., the circle whose diameter is AB.





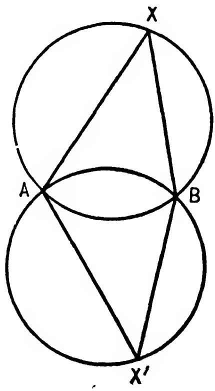
2. Every point X in line DE which joins the midpoints of two sides of an equilateral $\triangle ABC$ satisfies the following condition: The sum of the perpendiculars dropped from X upon the three sides is equal to the altitude of the triangle. Hence, if we did not examine the points without DE, we would consider DE as a locus, while actually it is no locus at all, since every point within the triangle ABC satisfies the above condition. If we admit negative distances, then every point in the entire plane satisfies the above condition.

Another way of proving a locus would be to show that:

1. Every point that satisfies the condition lies in the line.

2. The line contains no points which do not satisfy the condition.

Again the second part must not be omitted, since the line may contain all points satisfying the condition, and others that do not. Hence we may be led to consider a line as locus, only a part of which is really a locus.



If, for instance, A and B are two fixed points, every point X which satisfies the condition $\angle AXB = 45°$ lies in one of the two circles AXB and $AX'B$. But it would be erroneous to say the locus consists of the two circles, since there are points in the two circles which do not satisfy the condition, viz., all points in the minor arcs.

Hence — unless we consider a line as consisting of an infinite number

of points, and define a locus as the sum total of all points satisfying a condition — every proof of a locus involves the proofs of two theorems, and these two theorems stand in the relation of theorem and converse, or theorem and opposite. A concrete example will make this clearer. The fact that the perpendicular bisector of a line is the locus of a point equidistant from the ends of the line implies four theorems:

I. THEOREM	II. CONVERSE
Every point in the line is equidistant.	Every point that is equidistant lies in the line.
III. OPPOSITE	IV. CONVERSE OF OPPOSITE
Every point without is unequally distant.	Every point unequally distant is without the line.

A locus is true if all four theorems are true, and *vice versa*. But we saw in a preceding chapter * that these four theorems must be true, if any two adjacent ones are true, *i.e.* the locus is proved if we demonstrate I and II, or I and III, or II and IV, or III and IV. The first two combinations were discussed above, the last two are hardly ever used.

It is easy to see that the above facts are quite general. If we discard the two last combinations, we may say that each locus can be demonstrated in two different ways, and it is worth noticing that these two ways vary in difficulty for different exercises. To make a

* Chapter IX.

wise choice between the two methods we do not need to consider the first part of each combination, for they are identical, but we have to consider whether the converse or the opposite is easier to prove.

To demonstrate, for instance, that the bisector of an angle is the locus of a point equidistant from the sides of the angle, we have to prove either, that every point equidistant from the sides lies in the bisector, or that every point without the bisector is unequally distant from the sides. The first way, *i.e.* the proof of the converse, is easier in this case.

On the other hand, for proving that the locus of the vertex of a triangle whose base is a fixed line, and whose vertical angle equals a given angle, is a segment of a circle, the opposite leads to the result more easily than the converse.

Proofs simplified by the use of loci. — If a locus is proved, four theorems are established, and this fact is sometimes useful for proving theorems. Suppose we have proved the locus: "A plane perpendicular to a line at its midpoint is the locus of a point equidistant from the ends of a line," and we wish to demonstrate the theorem: "Three points not in a straight line, each equidistant from the end of a line, determine a perpendicular bisecting plane of the line." As a rule students will start an independent investigation for this theorem, and not see the connection between it and the preceding locus. Obviously each of the three points must lie on the perpendicular bisecting plane, according to the preceding locus, and since only one plane can be passed through the three points, this plane is the perpendicular bisecting plane.

Some difficult loci. — The most widely used loci are given in nearly all textbooks; others can be easily con-

structed, especially those that relate to the centers of circles which satisfy certain conditions, *e.g.* circles which touch a line at a given point, touch a circle at a given point, touch a circle and have a given radius, etc. Also proportional division of lines that radiate from a certain point and terminate in other lines yields many loci.*

A few difficult loci, which are valuable in solving difficult exercises, may not be without interest.

Let A and B be two fixed points, and x and y their respective distances from a third point X.

1. If $x^2 + y^2$ is a constant, the locus of X is a circle whose center is the midpoint of AB (proof by the median proposition).
2. If $x^2 - y^2$ is a constant, the locus of X is a straight line $\perp AB$, meeting AB at a point C so that $\overline{AC}^2 - \overline{CB}^2 = x^2 - y^2$.
3. If $\frac{x}{y}$ is a constant, the locus of X is a circle. The ends of a diameter of this circle are obtained by dividing AB internally and externally in the ratio x to y.

Let M and n be the perpendiculars dropped from a point X upon the sides of an angle ABC.

4. If $m + n$ is a constant, the locus of X is the perimeter of a rectangle; if $m - n$ is a constant, the locus consists of the prolongations of the sides of the same rectangle.
5. If $\frac{m}{n}$ is a constant, the locus of X is a straight line passing through B.

Pedagogic aspect of loci. — *When to teach loci.* — A great deal has been said and written about the importance of loci, and the insuffiency of the treatment of this topic in schools. The importance of loci in advanced mathematics appears to have influenced some writers,

* See page 247.

who seem to consider loci the most important subject in the geometry course, and wish to have this subject taught in a rigorous fashion at the very beginning of the work.

Important the subject undoubtedly is, still one ought not to go to the extreme of neglecting other topics on this account, and of introducing it at a time when the student's mind is not ready for it.

To give an exact presentation of the subject at the beginning of the first book can hardly be recommended, although the term "locus" may be mentioned in a provisional form when the theorem of the perpendicular bisector of a line, or of the bisector of an angle, is studied. It may then simply be stated that the locus is the "place," *i.e.* the line, in which a certain point must lie. An exact formal study it is probably better to postpone to the end of the second book. Scientifically, of course, some of the simplest problems, such as the construction of a triangle whose sides are given, depend upon loci, but such problems may be treated in the beginning quite satisfactorily without any reference to loci.

The teaching of the term "locus." — A formal definition of locus will of course not convey as clear a notion to the student as an explanation based upon numerous illustrations. Easy examples taken from mechanics show that under certain conditions a point must move or lie, in a certain line, or in a certain surface. Thus we may ask:

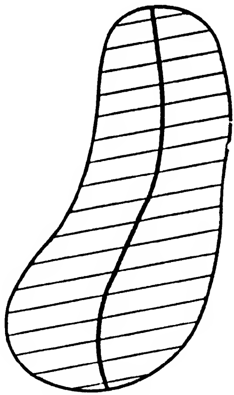
"Where must the center of a wheel lie, while the wheel moves on a straight track?

"If the locus is the place, or the line, in which a point must lie, what is the locus of the center of the wheel in the preceding question?

"What is the locus of one end of a stretched string, if the other end is fixed, and both ends are on the surface of a table? if only the moving end is on the table?"

Similarly, we may find the locus of a corner of a book when the book is opened, of a point of an elevator car while it moves, of a point of a locomotive while it is turned on a turntable, etc.*

Another class of exercises that may be used to give to the student a clear idea what a locus really is are drawing exercises. Let the student—perfectly empirically—draw exactly a large number of points satisfying a certain condition, and by joining these points find the locus. Thus he may draw empirically the locus of the mid-points of a set of parallel chords in any closed curve. Practically all loci relating to elementary geometry may be treated this way.



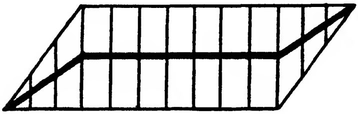
A few additional illustrations suitable for empirical drawing work are the following:

The locus of the ends of all tangents of a given length drawn to a given circle.

* The idea that a locus is generated by a moving point, or is the place where one point must lie, finds its expression in the phrase "locus of a point," which is possibly more widely used than "locus of points."

The locus of the feet of all perpendiculars drawn from one fixed point A, upon lines drawn through another fixed point B.

The locus of the centers of all circles of given radius touching a given circle.



The locus of the midpoints of all lines parallel to a given line and terminating in the perimeter of a given polygon, *e.g.* a parallelogram.

The locus of a point lying within a square (2×2 inches), and having a distance of $\frac{1}{2}$ inch from the perimeter, etc.

To discover loci, we use practically the same method as in the preceding paragraph. Of course the mere drawing is then no longer sufficient. The student has to determine what kind of a line the locus is, and has to prove his result. But the drawing will lead him in most cases to the result. If a few points are not sufficient to make the pupil discover the character of the locus, let him increase the number of these points, and if his drawing is exact, he will hardly fail to see what kind of a line the required locus is.

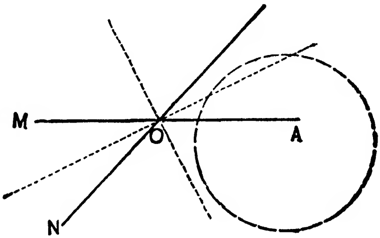
In addition to the six or seven which practically all textbooks contain, and which the student is expected to remember, a considerable number of others should be given merely for the purpose of giving the student facility in such work.

Application of loci. — Loci are used to find points which satisfy two conditions (in solid geometry sometimes three), and the great advantage of the locus method is that it enables us to consider each condition separately without paying any attention to the others.

Thus to find a point X, equidistant from two given lines M and N, and at a given distance d from a fixed point A, we have to consider the following two conditions:

1. X is equidistant from M and N.
2. The distance of X from A equals d.

Each condition leads to a locus, and it is advisable to distinguish these loci from each other, and from the other lines of the diagram, by some graphic distinction. Thus we may draw the lines that constitute the first locus in one color, and the second locus in a different color.



Or if no colored crayon is available, long-dotted and short-dotted lines may be employed. The two colors will diminish the danger of confusion in determining the points of intersection of the two loci. In most cases a "discussion" of the problem should be given, *i.e.*, we should determine the condition that would lead to no solution, or to one or to more solutions.

PUTTING GIVEN PARTS TOGETHER WITHOUT ANALYSIS

General remarks. — The simplest constructions of figures are those that can be accomplished by a simple putting together of the given parts without any previ-

ously devised plan.* To construct a triangle having two sides and the included angle is such a simple matter that no special analysis is necessary; we simply draw one given part, place the next one in its required position, etc. There is no difficulty in constructions of this type if attention is paid to one point, viz.: *The difficulty of the construction usually depends upon the choice of the part that is drawn first.*

It is easier to draw a right triangle having given the hypotenuse and an arm, by drawing at first the right angle or the arm, than by starting with the hypotenuse. To construct a quadrilateral having given the four sides and an angle, is a very simple matter if we commence with the angle or with one of the adjacent sides, but it becomes very difficult if we start with one of the other sides. In constructing a triangle having given two sides and the angle opposite one of them, we should begin with the angle or the adjacent side, but not with the opposite side, etc.

Fundamental constructions of triangle. — Typical illustrations of the method discussed in the preceding paragraphs are the fundamental constructions of triangles that may be referred to by the symbols *s.s.s.*, *s.a.s.*, *s.s.a.*, *s.a.a.*, *a.s.a.*, and hy. arm.† They derive a particular importance from the fact that a large percentage of problems finally depend upon the construction of triangles, and thereby upon one of these six problems.

* It was pointed out in one of the preceding paragraphs that the simple constructions of this type are, strictly speaking, based upon loci, but that a knowledge of loci is not necessary for understanding them.

† Hy. arm is of course only a special case of *s.s.a.*

It is very convenient to use for all triangle constructions a notation to which reference has been made before,* namely, that of designating the parts of a triangle as follows:

The sides	by a, b, c
The opposite vertices	by A, B, C
The corresponding angles	by α, β, γ †
The corresponding medians	by m_a, m_b, m_c
The corresponding altitudes	by h_a, h_b, h_c
The corresponding angle-bisectors	by t_a, t_b, t_c
The radius of circumcircle	by R
The radius of incircle	by r
The area of the triangle	by F
One half the perimeter	by s

It is worth noticing that a triangle is determined if three independent parts are given. Thus a triangle can be constructed if the three sides, or two sides and an included angle, or the three medians are given, for these parts are independent parts. But a triangle is not determined if the three angles are given, for the three angles are dependent, and really represent only two independent parts. Hence an infinite number of triangles exist which contain the given angles. A few other illustrations of dependent parts are the following: b, h_a, C; a, A, R; A, $s-a$, r.

Similarly a quadrilateral is determined by 5, a polygon

* See page 208.

† If Greek letters are objected to, we may use the letters A, B, C, although this sometimes leads to ambiguities.

of n sides by $2n - 3$, independent parts. It should be borne in mind, however, that a problem is not called indeterminate because it has several solutions. A problem is *indeterminate* if it has an infinite number of solutions, *determinate* if the number of its solutions is finite. Thus a problem that has four different solutions is a determinate problem.

A problem that has no solution is an impossible problem. All problems that are indeterminate on account of the dependence of the given parts become impossible if the given parts do not conform to the relation of dependence. Thus no triangle can be drawn containing three given angles if the sum of these angles is not equal to 180°.

GEOMETRIC ANALYSIS OF SIMPLE PROBLEMS

General description. — All problems that cannot be solved by a direct putting together of given parts or by means of loci should be attacked by analysis. Many different general presentations of the mode of analyzing a problem have been given, but most of them are too abstract to be of great service to the beginner. The author has attempted to give, in the following, a description of an analysis as elementary and concrete as possible, although it loses thereby some of its generality.

1. Make a diagram resembling the one required but not necessarily having the same dimensions.

2. Determine (*a*) all lines, (*b*) all angles, that are directly given, or that can be easily found from the given parts, and mark all these parts.

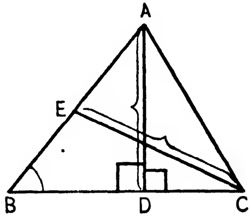
3. Examine all the triangles of the diagram until you discover one that can be constructed.

4. Make this triangle the basis of the construction, and try to determine successively all other points of the figure.

5. In case no triangle can be found that can be constructed directly, draw additional lines which will enable you to obtain such a triangle.

A concrete illustration. — To obtain a clearer understanding of the meaning and scope of these rules let us consider a concrete example, viz.: To construct a triangle having given β, h_a, and m_c.

1. To obtain a figure resembling the required one, we draw any triangle ABC, AD the altitude from A, and CE the median from C.



2. If triangle ABC were the required one, we should know:

(*a*) $AD\,(=h_a)$, and $CE\,(=m_c)$.
(*b*) $\angle B\,(=\beta)$, $\angle ADC\,(=90°)$, and $ADB\,(=90°)$.

3. If we examine the various triangles, we find that ADB contains one known side and two known angles, and hence can be constructed.

4. After this triangle ADB is constructed, we can determine E, the midpoint of AB, and then C, for the length of CE is given.

Discussion of the parts of an analysis.

1. *Drawing of diagram resembling the required one.* — Obviously, the average student, in attacking a com-

plex problem, has no clear conception of the given parts, of the required parts, and of their relations to each other, until he sees a diagram representing all these things. Every architect or engineer, in solving a practical problem, will make at first a rough sketch that will enable him to see clearly the relation of the various points to be considered, and only after the problem is solved, will he make an exact diagram of the proper dimensions. Similarly the student of geometry needs a sketch to make clear to him the real nature of the problem. Of course such a sketch will not necessarily have the same dimensions as the required figure, and it would be foolish even to attempt to draw it so.

2. *Determination of known parts.* — Let the student systematically examine all lines and all angles of the diagram, and let him determine those that are known. In doing so he may include a few that are not necessary for the construction. But, as a rule, it is advisable for the beginner to be systematic in this work, since his chances for finding a solution are increased thereby.

The determination of the parts that are not given directly, but that can be found, is a matter of great importance, but as this subject is rather complex it will be discussed in a special section.*

3. *Discovery of a triangle that can be constructed.* — To look for a *triangle* as the basis of a construction is, of course, an arbitrary limitation, as it may be a point, or a square, or a circle, or other figure that can be drawn first. In most cases, however, it is a triangle that can

* See page 233.

be used as the formation of the construction. A systematic survey of *all* triangles of the diagram can hardly fail to make the student discover the proper triangle.

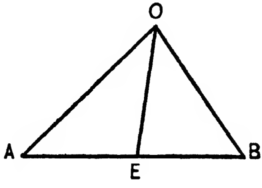
4. *Make the triangle the basis of the construction.* — After the initial triangle is constructed, it is usually a simple matter to find the rest of the figure. A few suggestions may prove helpful in complex cases.

(*a*) Place the diagram of the construction, as far as possible, in a position similar to the diagram of the analysis; much confusion is thereby avoided.

(*b*) Use the same letters for designating the construction as for the analysis. As students only rarely hand to the teacher a written account of an analysis, no misunderstanding can arise from such a practice. If you expect your students to write out the analysis, then use A', B', C', etc., for A, B, C, etc.

(*c*) In many examples the student has a choice of several ways for completing the construction. In such a case select the construction that is easiest of proof.

Suppose, for instance, the student, in constructing a triangle having given m_a, m_b, and m_c, has constructed $\triangle OAB$ so that $OA = \frac{2}{3} m_a$, and $OB = \frac{2}{3} m_b$, and $OE = \frac{1}{3} m_c$. He may then complete the figure either by producing EO by twice its length and joining, or by producing AO and BO, by one half their lengths and joining to A and B respectively.* The first way, however, gives a much



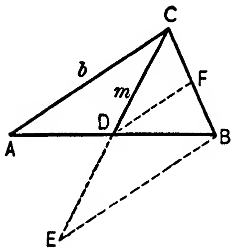
* There are, of course, still other ways of completing the figure.

simpler proof than the second, hence it should be selected.

5. *Drawing additional lines.* — In case we cannot find any triangle that can be constructed, we have to draw additional lines which produce such triangles.

The general principles that underlie this procedure are rather complicated, and will be treated in the section on difficult analysis.* Here only a few comparatively simple illustrations may find place.

To construct a triangle having given a, b, and m_c. If ABC were the required Δ, we should know $CB\ (=a)$, $CA\ (=b)$, and $CD\ (=m_c)$. Hence, no triangle can be constructed. But if we produce CD by its own length to E and draw EB, we can construct $\Delta\ CEB$, for we know its three sides.



Or we may bisect CB, and join its midpoint F to D, then we can construct $\Delta\ CDF$, for $CD = m_c$, $CF = \frac{a}{2}$, and $DF = \frac{b}{2}$.

Or we may produce AC by its own length to G and draw GB. Then $\Delta\ BCG$ can be constructed, since $CG = b$, $CB = a$, and $BG = 2\,m_c$.

If a sum or difference of two lines (or angles) is given, it is usually necessary to construct that sum or difference in the analysis; thus if $a + b$ is given, produce a by the length b or b by the length a. If $a - b$ is given, lay off b on a, or a on b.

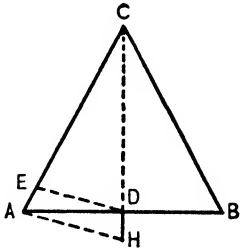
Thus, to construct an equilateral triangle, having given the difference of side and altitude, we may on CA lay off $CE = CD$ and

* See page 239.

draw DE. Then $\triangle AED$ can be constructed, since we know one side (EA) and all angles.

Or we may on CD lay off $CH = CA$ and draw AH. Then $\triangle ADH$ can be constructed.

For further illustrations of the preceding methods, and exercises based upon them, the reader is referred to Schultze and Sevenoak's Geometry.



GEOMETRIC ANALYSIS OF DIFFICULT PROBLEMS

The principles of the preceding section are sufficient to solve the large majority of the problems that occur in secondary schools, and hence they constitute all that is necessary for such schools. For the teacher, however, it is of great importance to obtain a more complete mastery of the subject, and hence additional methods will be given here. They consist of further elaboration of parts two and five of the analysis as given above, viz.:

(*a*) Method for determining lines and angles of the diagram, and

(*b*) Methods for drawing additional lines, which will lead to triangles that can be constructed.*

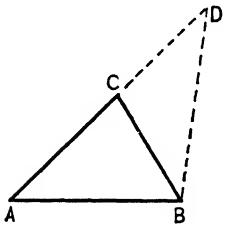
Methods for determining the known parts of a diagram. — To discover in an analysis all parts of the diagram that are not given directly, but that may be found indirectly, the student has to be acquainted with regular propositions of geometry. Thus, he has to know that the diagonals of a parallelogram bisect each other,

* The most important book on methods of attack is the work of Petersen, Methods and Theories, Kopenhagen, originally written in Danish, but translated into German (Kopenhagen) and French.

that the sum of the angles of a triangle is 180°, that the opposite angles of an inscribed quadrilateral are supplementary, that the diagonals of a rhombus are perpendicular to each other, etc. If, for instance, in a triangle $\alpha+\beta$ is given, γ can be determined. If in an isosceles triangle one interior or exterior angle is known, all other interior or exterior angles may be determined, etc.

But in addition to these well-known geometric facts, a number of relations that exist in certain diagrams are of great importance for successful analyzing. Some of the most important ones are the following:

1. If in $\triangle ABC$, AC is produced to D so that $CD = CB$, then in $\triangle ABD$:



$$AB = c.$$
$$AD = a + b.$$
$$\angle A = \alpha.$$
$$\angle D = \frac{\gamma}{2}.$$
$$\angle ABD = 90° + \frac{\beta - \alpha}{2}.$$

The altitude from $B = h_b$.

Hence, if of the parts c, $a+b$, α, γ, $\beta-\alpha$, and h_b any three independent * ones are known, $\triangle ABD$ can be constructed and the remaining three parts can be determined.

In some analyses involving $a+b$ it would be better to produce BC by the length CA, thus making a triangle that contains: $a+b$, c, β, $90° + \frac{\alpha-\beta}{2}$, $\frac{\gamma}{2}$, and h_a.

* Of the twenty possible combinations, only two give dependent parts, viz., c, α, h_b, and α, γ, $\beta-\alpha$.

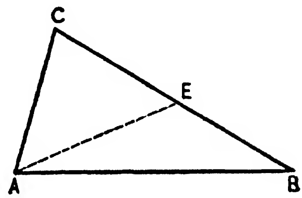
EXERCISES

Construct $\triangle ABC$, having given:

1. $a+b, \alpha, \gamma$.
2. $a+b, c, \alpha$.
3. $a+b, c, \gamma$.
4. $a+b, c, h_b$.
5. $a+b, c, \alpha-\beta$.
6. $a+b, \beta, \gamma$.
7. $a+b, \alpha, h_b$.
8. $a+b, \gamma, \alpha-\beta$.
9. $a+b, \gamma, h_b$.
10. $h_b, \alpha-\beta, \gamma$.
11. $a+b, \alpha-\beta, h_a$.

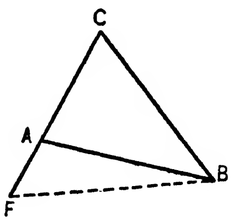
2. If in $\triangle ABC$ on CB we lay off $CE = CA$, then in $\triangle ABE$,

$$AB = c.$$
$$BE = a - b.$$
$$\angle B = \beta,$$
$$\angle BAE = \frac{\alpha-\beta}{2}.$$
$$\angle AEB = 90^\circ + \frac{\gamma}{2}.$$



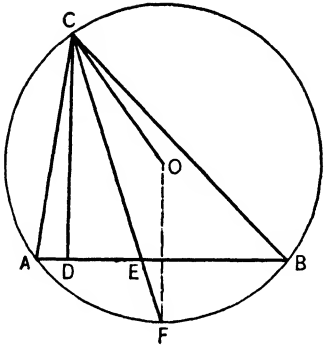
The altitude from $A = h_a$.

Hence, if of the parts c, $a-b$, β, $\alpha-\beta$, γ, and h_a any three independent parts are known, the $\triangle AEB$ and the other three parts may be constructed.



If we produce CA to F, so that $CF = CB$, then triangle ABF is determined by any combination of three independent elements that can be formed from the elements: c, $a-b$, α, $\alpha-\beta$, γ, and h_b.

Exercises relating to this diagram may be obtained by substituting minus signs for plus signs in the preceding set of exercises.



3. If in $\triangle ABC$, the altitude CD or h_c, the angle-bisector CE or t_c, and the radius of the circumcircle CO or R be drawn, then

$$\angle DCE = \angle ECO = \frac{\alpha - \beta}{2}.$$

Triangle CDE can be constructed if there are given h_c and t_c, or h_c and $\alpha - \beta$, or t_c and $\alpha - \beta$.

It is worth remembering that the perpendicular bisector of AB passes through O and bisects arc AB in F, and that CE produced also passes through F.

EXERCISES

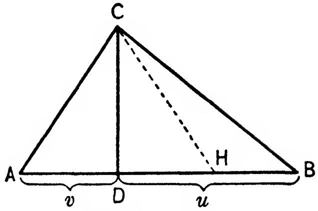
Construct $\triangle ABC$, if there are given:

12. $h_c, \alpha - \beta, a$.
13. $h_c, \alpha - \beta, R$.
14. $h_c, \alpha - \beta, m_c$.
15. $t_c, \alpha - \beta, b$.
16. $t_c, \alpha - \beta, \gamma$.
17. $t_c, \alpha - \beta, R$.
18. $t_c, \alpha - \beta, m_c$.
19. t_c, h_c, a.
20. t_c, h_c, α.
21. h_c, t_c, R.
22. h_c, t_c, m_c.

4. In triangle ABC let

CD or h_c be the altitude upon c,
BD or u be the projection of a upon c,
AD or v be the projection of b upon c,

and make $DH = AD$.



Then in triangle BCH, $BC=a$, $HB=u-v$, $CH=b$, $\angle B=\beta$, $\angle CHB=180^\circ-\alpha$, $\angle HCB=\alpha-\beta$, and the altitude from $C=h_c$. Hence, if of the parts a, b, $u-v$, h_c, α, β, $\alpha-\beta$, any independent three are known, $\triangle CAB$ can be constructed.

EXERCISES

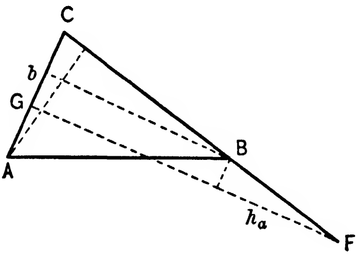
Construct $\triangle ABC$, if there are given:

23. $\alpha-\beta, a, b$.

24. $\alpha-\beta, u-v, a$.

25. $\alpha-\beta, u-v, h_c$.

26. $\alpha, \beta, u-v$.



5. If in triangle ABC, CB is produced to F so that $BF=b$, and FG is drawn perpendicular to CA, then in $\triangle CFG$, $CF=a+b$, $FG=h_a+h_b$, $\angle C=\gamma$, and $\angle CGF=90^\circ$.

Similarly, a right triangle can be obtained that contains $a-b$, h_b-h_a, and $\angle C$.

EXERCISES

Construct $\triangle ABC$, having given:

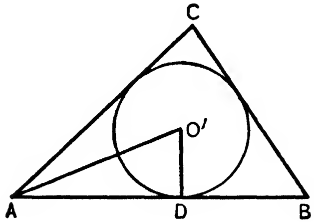
27. h_a+h_b, b, γ.

28. h_a+h_b, a, b.

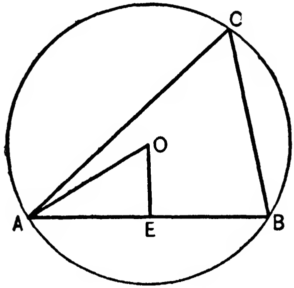
29. h_a-h_b, a, γ.

30. h_a-h_b, a, b.

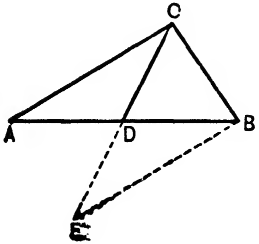
A few additional cases may be mentioned briefly.



6. If O' is the incenter of $\triangle ABC$, D a point of contact, then $O'D = r$, $AD = s - a$, and $O'AD = \frac{\alpha}{2}$.



7. If O is the circumcenter of $\triangle ABC$, and $OE \perp AB$, then $AE = \frac{c}{2}$, $\angle AOE = \gamma$, and $OE = R$.



8. If a median CD of $\triangle ABC$ is produced by its own length to E, then in $\triangle CEB$, $CB = a$, $BE = b$, $CE = 2\ m_c$, $\angle CBE = 180° - \gamma$, the altitudes from E and C equal respectively h_a and h_b.

EXERCISES

Construct $\triangle ABC$ if there are given:

31. m_c, a, γ.
32. $s - a$, α, β.
33. $b + c$, a, r.
34. R, α, m_a.
35. m_c, a, h_b.

Methods for drawing additional lines. — The fact that in certain problems no triangle can be found that can be constructed directly, is usually due to the circumstance that the known parts do not lie together. Consequently, we have to bring them together, and any method that will accomplish this may be employed. There are three methods of moving figures that are most useful. These methods, already mentioned in the chapter on inequalities,* are the following ones:

I. Translation.

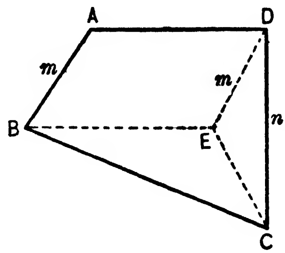
II. Rotation about a point.

III. Rotation about a line.

The application of these methods is best explained by concrete examples.

I. **Translation.**

1. To construct a quadrilateral $ABCD$, having given its four angles (α, β, γ, δ) and two opposite sides ($AB = m$, $CD = n$).

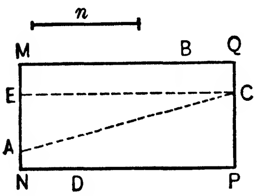


Analysis. — If $ABCD$ were the required quadrilateral, we should know its four angles, and the sides AB and CD. As no triangle can be constructed, we translate AB into the position DE, *i.e.* we make DE equal and parallel to AB. Since $\angle ADE = 180° - \alpha$, we know in $\triangle DEC$ two sides and the included

* See page 164.

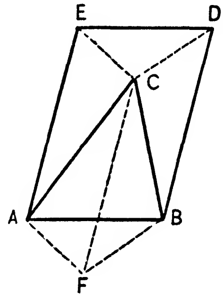
angle, viz., $ED = m$, $DC = n$, and $\angle EDC = \delta - (180° - \alpha) = \alpha + \delta - 180°$. Hence, $\triangle EDC$ can be constructed.

2. To construct a rectangle so that each side passes through one of four given fixed points A, B, C, and D, and that one side of the rectangle equals a given line n.



Analysis. — If $MNPQ$ were the required rectangle we should know MQ $(= n)$ and all lines joining any two of the given points. But no triangles, except those formed by A, B, C, and D, can be constructed. Hence, translate MQ into the position EC, then EC $(= n)$, $\angle AEC$ $(= 90°)$, and AC are known. Hence $\triangle ACE$ can be constructed.

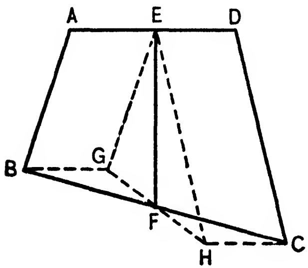
3. On the base AB of $\triangle ABC$ to construct a parallelogram $ABDE$ so that CE and CD equal two given lines.



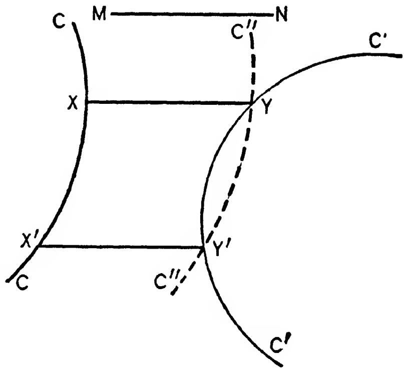
Analysis. — If $ABDE$ were the required parallelogram, we could construct $\triangle ECD$, since we know its three sides. But we cannot place it in the correct position. Hence, translate $\triangle ECD$ into the position ABF. Here it can be constructed, and by making AE and BD parallel to FC we may translate it into the required position.

4. To construct a quadrilateral having given its four sides, and the line joining the mid-points of two opposite sides.

Analysis. — If $ABCD$ were the required quadrilateral, we should know AB, BF, FC, CD, DE, EA, and EF. Since the parts do not form triangles, draw EG equal and parallel to AB, and EH equal and parallel to DC. Since triangles BGF and CHF are equal, GFH is a straight line and $GF = FH$. Hence, $\triangle GEH$ can be constructed, having given two sides and the median to the third side.



5. To draw a line equal and parallel to a given line MN which has one end in the curve C, the other in the curve C'.



Translate the curve C by the distance MN into the position C''. Then any line drawn from a point in C'' parallel to MN and terminating in C will be equal to MN. The required line XY is drawn from Y, the point of intersection of C'' and C'.

Since in place of the curves we may have straight lines, the perimeter of triangles or other figures, circumferences, etc., a large number of concrete cases are covered by this example.

EXERCISES

1. To construct a trapezoid, having given its four sides.

2. To construct a trapezoid, having given the two diagonals and the two parallel sides.

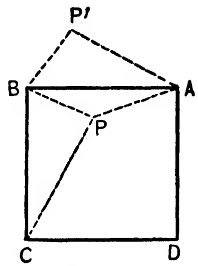
3. To construct a quadrilateral, having its four sides and the angle formed by the prolongations of two opposite sides.

4. To construct a quadrilateral, having given two opposite sides, the diagonals, and the angle formed by the diagonals.

5. To construct a quadrilateral $ABCD$, having given the angles A and B, the diagonals AC and BD, and the angle formed by the diagonals.

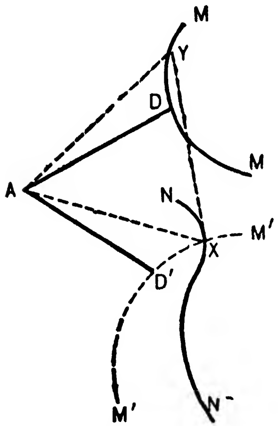
II. Rotation about a point.

1. To construct a square $ABCD$, having given the distances of a point P within from the vertices A, B, and C.



Analysis. — If $ABCD$ were the required square, we should know PA, PB, and PC, and the angles A, B, C, and D. Rotate $\triangle BPC$ about B through an angle of 90°. Then $\angle P'BP = 90°$ and $BP' = BP$. Hence $\triangle BP'P$ can be constructed, and since $P'A = PC$, $\triangle P'PA$ can also be drawn, etc.

2. To construct an isosceles $\triangle ABC$ so that the vertical angle A equals a given angle α, that A has a fixed position, and that B and C lie respectively on two given curves* M and N.

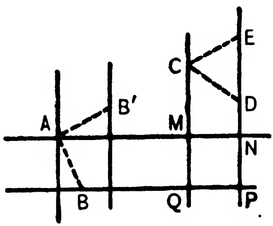


Solution. — Rotate curve M about A through an angle equal to α. By this rotation any line from A to M, as AD', takes a position AD' which forms with AD an angle equal to α. Or any two lines drawn from A, including an angle equal to α, and ending in M and M' respectively are equal. If X is the intersection of M' and N, then draw AX and AY so that $\angle XAY = \alpha$, and $\triangle AXY$ is the required triangle.

A combination of translation and rotation is used in the following problem.

3. To construct a square $MNPQ$ so that two opposite sides pass through two given points A and B, and the other two opposite sides pass through the given points C and D.

Analysis. — Since no triangle can be constructed, rotate AB, AN, and BP about A through an angle of 90°.



* For the sake of brevity the term "curves" is used here and on the following pages, when curves, straight lines, broken lines, in fact all kinds of lines, are meant.

If B' is the new position of B, our problem would be to draw four parallel lines through A, B', C, and D, so that the distance of the first pair equals the distance of the second pair. This is done by translating AB' into the position CE.* Our two pairs of parallels become then identical. Hence we can draw $\triangle\ CED$.

EXERCISES

1. To construct a regular hexagon $ABCDEF$, having given the distances of a point within from A, B, and C.

2. Compute the area of a square $ABCD$ if the distances from a point within from A, B, and C are respectively two, three, and four. (Rotate two ⚠.)

3. Construct a right isosceles triangle so that the vertex of the right angle takes a fixed position and the other vertices lie in two given circumferences.

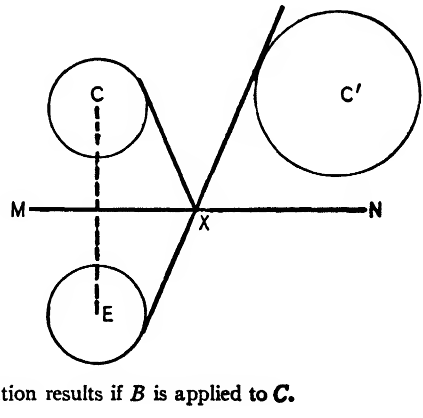
4. To construct an equilateral triangle so that its vertices lie respectively in three given parallel lines.

5. To construct an equilateral triangle so that its vertices lie in three given concentric circumferences.

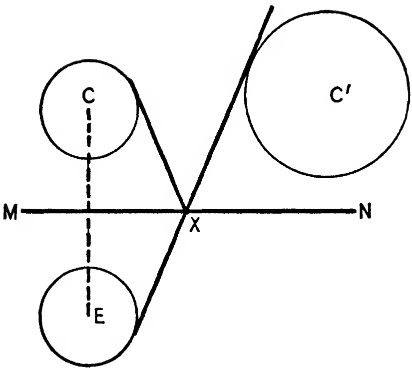
6. Given quadrilateral $ABCD$ and isosceles $\triangle\ MNP$, with $MP = NP$. To construct a $\triangle\ EFG \sim PMN$ so that E coincides with A, F lies in BC, and G lies in CD.

III. Rotation about a line (turn about an axis).

1. Given two circles C and C' on the same side of a line MN. Required a point X in MN so that the two (inner) tangents, drawn from X to C and C' respectively, form equal angles with MN.



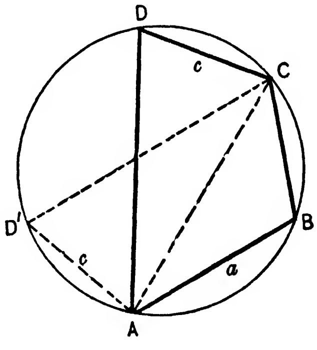
* A second solution results if B is applied to C.



Solution. — Turn the figure about MN as an axis so that C takes the position E. Draw a common interior tangent to C and E. This tangent intersects MN in X.

A special case of this rotation about a line is the turning of a figure so that the ends of a line are interchanged, *i.e.* A, one end of a line, is put in the position formerly occupied by B, and B in the position formerly occupied by A. (Similarly for angles.)

2. In a given circle to inscribe a quadrilateral, having given two opposite sides AB and CD (or a and c) and the sum of the other two sides ($AD + BC$ or s). Suppose $ABCD$ were the required quadrilateral. Since no two known sides lie together, we turn $\triangle DCA$ so that A and C exchange positions, and D takes the position D'. Then $\triangle D'AB$ can be easily constructed, and afterward $\triangle D'BC$ can be found, having given the base $D'B$, the opposite angle, and the sum of the other two sides.



Having thus obtained $ABCD'$, we find it a very simple matter to construct $ABDC$.

EXERCISES

1. Construct quadrilateral $ABCD$ if the four sides are given and $\angle ADB = \angle CDB$.

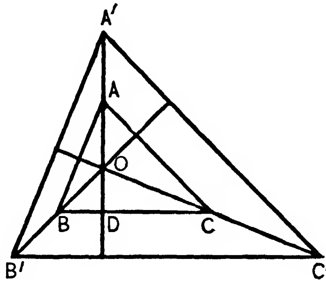
2. Given two points P and Q on the same side of a straight line MN. Required a point X in MN so that $PX + QX$ is a minimum

3. To construct a square so that two opposite vertices lie in two given circumferences, and the other two vertices lie in a given straight line.

4. In a given circle to inscribe a quadrilateral so that one side equals twice its opposite side and the two other opposite sides equal given lines.

SPECIAL DEVICES

Method of similarity. — If a required figure contains only one given line, we may at first discard this line, and construct a figure which is similar to the required one, and then change the dimensions of the figure so as to contain the given line.*

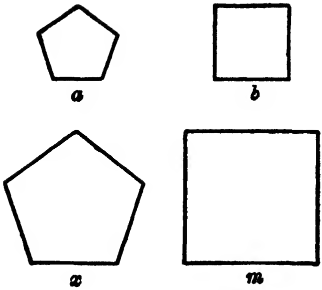


1. To construct a triangle, having given the angles and the upper segment of h_a.

Construction.—Construct any triangle ABC containing the given angles, and draw its altitudes. Produce OA to A' so that OA' equals the given upper segment, and then build up the required $\triangle A'B'C'$ by means of several similar triangles.

Frequently this method enables us to solve a problem, if we can solve the opposite one. Thus, we can inscribe in a given semicircle a square, if we can circumscribe a semicircle about the square; we can transform a square into an equilateral triangle, if we can transform an equilateral triangle into a square, etc.

* The method may be extended to many cases containing two given lines. Construct then at first a figure in which the two lines have an arbitrary length, but the correct ratio, and then change the dimensions.



2. To transform a square into a regular pentagon.

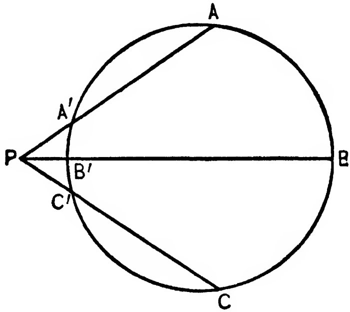
Construction. — Construct any regular pentagon and transform it into a square. Let a be the side of the pentagon, b the side of the equivalent square, m the side of the given square, and x the side of the required pentagon.

Obviously $\quad b^2 : m^2 = a^2 : x^2.$

Or $\quad b : m = a : x.$

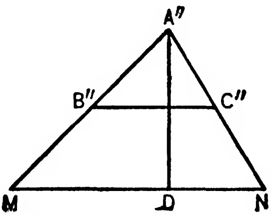
I.e., x is the fourth proportional to b, m, and a.

3. To construct a triangle, having given its three altitudes (h_a, h_b, h_c).



Solution. — Since $ah_a = bh_b = ch_c$, we can easily find three lines that are proportional to a, b, and c. From any point P draw three secants PA, PB, and PC respectively equal to h_a, h_b, and h_c. Then the exterior segments of these secants PA', PB', PC' are proportional to a, b, and c.

Hence, a triangle $A''B''C''$, whose sides are equal respectively to PA', PB', and PC', is similar to the required one. If the altitude



from A'' is produced to D so that $A''D = h_a$, and MN is drawn parallel to $B''C''$, then $\triangle A''MN$ is the required one.

EXERCISES

Construct a triangle, having given:

1. α, β, m_a.

2. α, β, $b - h_a$.

3. α, β, $m_b - h_b$.

4. Construct a triangle, having given the angles and the distance of circumcenter from incenter.

5. Transform a square into an equilateral triangle.

6. Transform a square into a triangle similar to a given triangle.

7. In a given quadrilateral $ABCD$ to inscribe a rhombus whose sides are parallel to the diagonals of $ABCD$.

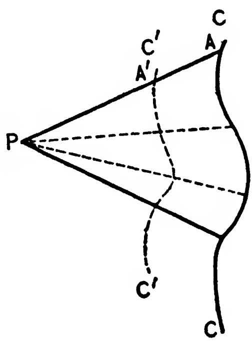
8. Inscribe a square into a semicircle.

9. In a given circle to inscribe a rectangle similar to a given rectangle.

10. In a given triangle to inscribe a parallelogram similar to a given parallelogram.

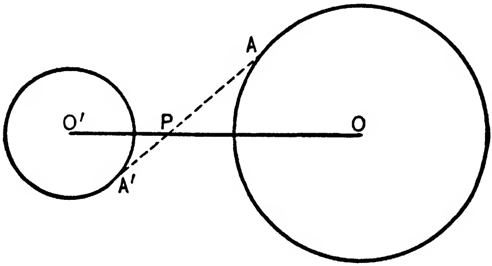
Multiplication of curves.

Definitions. — If from a point P a straight line is drawn to a point A in the curve C, and A' divides PA in a given ratio, then the locus of A' is a curve C', which is similar to C. The curves C and C' are said to be *radially situated* with respect to P, P is the *center of similitude*, and the ratio $\dfrac{PA'}{PA}$ is called the *ratio of similitude.*



If the point A' lies in the prolongation of AP, the result is similar to the one above. The ratio of simili-

tude, however, is negative, *e.g.*, in the annexed diagram $\frac{PA'}{PA} = -\frac{1}{2}$. In general, if the ratio of similitude is negative, P lies between the two curves.



The construction of the curve C' when the ratio of similitude is $\frac{m}{n}$, and the center of similitude is P, is sometimes referred to as *multiplying* C *by* $\frac{m}{n}$ *with respect to* P.

Thus, in the first diagram C is multiplied by $\frac{2}{3}$, in the second the circle is multiplied by $-\frac{1}{2}$ with respect to P.

This multiplication may be also applied to straight lines, broken lines, in fact to any figures whatsoever, and some of the results obtained are the following:

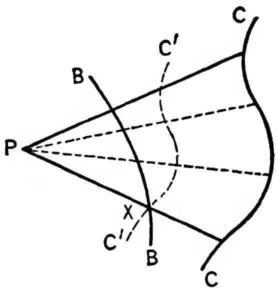
1. If a straight line L is multiplied by $\frac{m}{n}$, the result is another straight line L', and L and L' are parallel.

To construct L' it is necessary to find *one* point of L'.

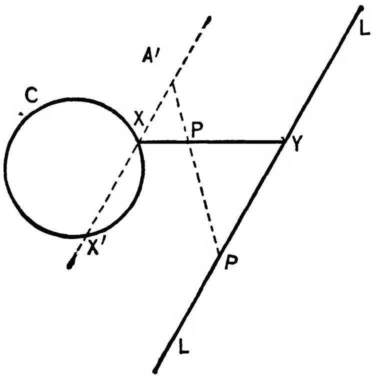
2. If any figure is multiplied by $\frac{m}{n}$, we obtain a simi-

lar figure. Homologous angles are equal, and the ratio of any two homologous lines is equal to $\frac{m}{n}$.

3. If a circumference is multiplied by $\frac{m}{n}$, the result is again a circumference. If O and O' are respectively the centers, and r and r' the respective radii, then $\frac{PO'}{PO}=\frac{m}{n}$ and $\frac{r'}{r}=\frac{m}{n}$. Hence the required circle can easily be constructed by first finding its center, and then its radius.



Applications. — The multiplication of curves solves the general problem: "From a fixed point P to draw a line so that the segments made on it by two given lines* B and C shall bear a given ratio $\frac{m}{n}$."

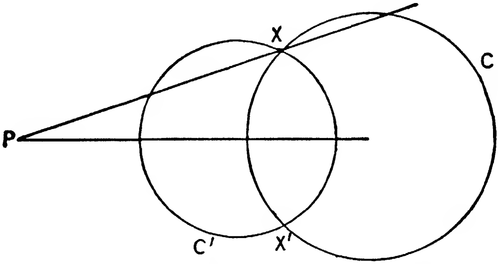


Multiply C by $\frac{m}{n}$ and let the resulting line C' intersect B in X. Then PX produced is the required line.

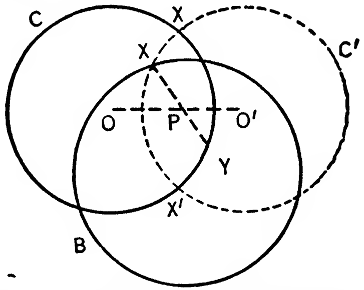
Thus, draw a line XY from a circumference C to a line L, passing through a point P, so that $XP:PY = 2:3$, multiply L by $-\frac{2}{3}$

* *I.e.*, curved, straight, broken, etc., lines.

with respect to P,* and let the resulting line intersect C in X and X'. XP and $X'P$ produced are the required lines.



From a point P without a circle C to draw a secant which shall be to its external segment as $4:3$, multiply C by $\frac{3}{4}$ with respect to P, and let the resulting circle C' intersect C in X and X'. PX and PX' are secants required.



To draw a line XY through P and terminating in two circumferences C and B, so that $XP = PY$.

Multiply C by -1 with respect to P, and let the resulting circle C' intersect B in X. XP produced is the required line.

EXERCISES

1. Multiply a line C by $-\frac{2}{3}$, by 2.
2. Multiply a circle by -2 if P lies in the circumference.
3. Multiply a circle by $\frac{3}{4}$ if P is within.

* This is easily done by drawing AP, producing it by $\frac{2}{3}$ of its length to A', and drawing $A'X \parallel L$.

4. Multiply a circle by $\frac{m}{n}$, if m and n are two given lines.

5. From a point P within an angle draw a line meeting the side of the angle in X and Y so that $PX : PY = 3 : 2$.

6. From a point without an angle draw a line meeting the sides of the angle in X and Y so that $\frac{PX}{PY} = 1:2$.

7. A point P and a circumference C lie on opposite sides of a straight line L. Through P to draw a line that meets L in X and C in Y so that $\frac{PX}{PY} = \frac{2}{3}$.

8. From a point P without a hexagon to draw a line that meets the perimeter in X and Y so that $PX : PY = 2 : 3$.

9. To draw a line terminated by two circumferences and bisected by a given point P.

10. Through a point within a circle to draw a chord whose segment shall be as m to n, when m and n are two given lines.

11. Through a point of intersection of two circles to draw a line so that the chords formed are as $2 : 3$.

12. A point P lies in the minor arc made by a chord AB. From P to draw a chord PQ intersecting AB in R so that $PR : RQ = 2 : 1$.

13. Given a quadrilateral $ABCD$ and a point P within. To construct a parallelogram whose center is P and whose vertices lie respectively in the sides of $ABCD$.

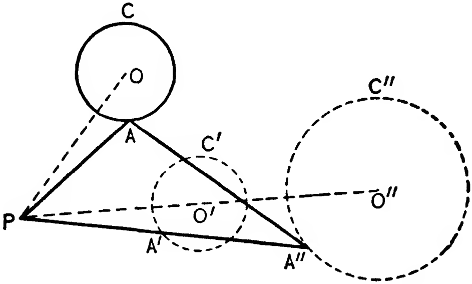
14. To construct (by the above method) a triangle, having given a, b, m_c.

15. Construct a triangle, having given a, α, m_b.

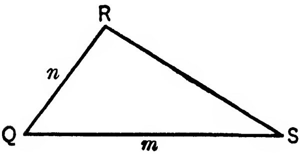
16. Construct a triangle, having given m_a, m_b, γ.

Multiplication and rotation. — If a curve C is rotated about a point P through an angle equal to a given angle Q, then any two lines PA and PA' which include an angle equal to Q and which terminate in C and C' are equal. If we now multiply C' by $\frac{m}{n}$ with respect

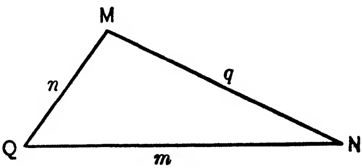
to P and denote the resulting line by C'', then any two lines PA'' and PA, which include an angle equal to Q



and terminate in C'' and C, have the ratio $m : n$. Or all the triangles PAA'' that have A in C, A'' in C'', and $\angle APA'' = \angle Q$ are similar. Hence, we may consider C'' the locus of the third vertex of a triangle that has one vertex in C, another in P, and that is similar to a given triangle QRS.



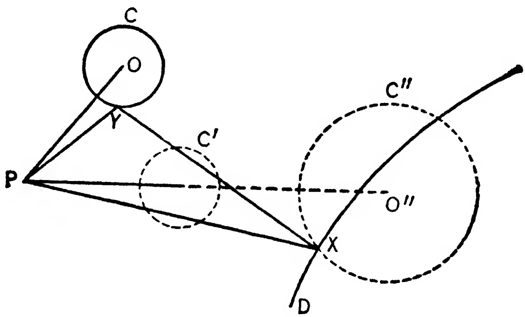
These considerations enable us to solve the problem: To construct a triangle similar to a given $\triangle mnq$, having one vertex in a given point P, another in a line* C, and a third in a line D.



Solution. — Rotate C about O through an angle equal to Q, and multiply the resulting line by $\frac{m}{n}$.

* The term "line" refers to any line, *i.e.*, a line that is straight, or not straight.

The intersection of the line C'', thus obtained, with D gives X, one vertex of the required triangle. By making $\angle XPY = \angle Q$ we obtain the third vertex Y.



NOTE. — The line C'' can often be obtained without constructing C'. Thus, in the above diagram, we could obtain O'' by making $\triangle OPO'' \sim \triangle MPN$, and r'' equal to the fourth proportional to n, m, and r.

EXERCISES

To construct a triangle similar to a given triangle MNQ, having the vertex corresponding to Q at a fixed point P, and the other two vertices respectively in

1. Two parallel lines.
2. Two intersecting lines.
3. A line and a circle.
4. The perimeter of triangle and a line.
5. Two concentric circles.
6. Two excentric circles.
7. To construct a triangle whose sides are as 3 : 4 : 5 and whose vertices lie in three concentric circles.
8. To construct a triangle whose sides are as 4 : 5 : 6 and whose vertices lie in three parallel lines.

ALGEBRAIC ANALYSIS

Algebraic analysis. — Most problems of geometry can be solved by means of algebra, but this mode of attack in elementary work is not as interesting as the purely geometric ones, since it requires very little originality, and since the constructions thus obtained frequently lack elegance and clearness. For the teacher, however, who sometimes is obliged to get a solution of a difficult problem in a very short time, it is a useful tool.

In an algebraic analysis the solution is usually made to depend upon the length of some particular unknown line or lines. The relations of this line or these lines with the known quantities are expressed by equations, whose solutions express algebraically the unknown line in terms of known quantities. The construction of this expression leads to the solution.

To apply this method the student has of course to be familiar with the construction of certain fundamental algebraic expressions, *e.g.*,

$$\frac{ab}{c},\ \sqrt{a^2+b^2},\ \sqrt{ab},\ a\sqrt{7},\ \sqrt{a^2-b^2},\text{ etc.,}$$

where a, b, c, are known lines.

Many complex algebraic expressions can be reduced to fundamental ones, *e.g.*,

$$\frac{abc}{de}=\frac{a\cdot\frac{bc}{d}}{e}.$$

$$\frac{ab+cd}{e}=\frac{ab}{e}+\frac{cd}{e}.$$

$$\frac{a^2-b^2}{c}=\frac{(a+b)(c-b)}{c}.$$

$$\sqrt{a^2-bc}=\sqrt{a^2-(\sqrt{bc})^2}.$$

$$a\sqrt[4]{3}=\sqrt{a\cdot\sqrt{a(3a)}}.$$

$$\sqrt[4]{abcd}=\sqrt{\sqrt{ab}\sqrt{cd}}.$$

$$\sqrt[4]{a^4+b^4}=\sqrt{a\sqrt{a^2+\left(\frac{b^2}{a}\right)^2}},\text{ etc.}$$

The general character of the method will be illustrated by two examples only, and the reader who wishes further details is referred to Schultze and Sevenoak's Geometry.

1. To divide a line AB in extreme and mean ratio.

Let $AB=a$, and the greater part of the divided line $=x$.

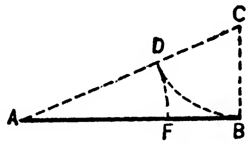
Then $$a:x=x:a-x.$$

$$x^2+ax=a^2.$$

$$x^2+ax+\left(\frac{a}{2}\right)^2=a^2+\left(\frac{a}{2}\right)^2.$$

$$x+\frac{a}{2}=\sqrt{a^2+\left(\frac{a}{2}\right)^2}.$$

$$\therefore\ x=\sqrt{a^2+\left(\frac{a}{2}\right)^2}-\frac{a}{2}.$$



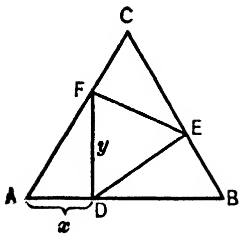
To construct this expression we draw $CB\perp AB$ and equal to $\frac{a}{2}$,

then $$AC=\sqrt{a^2+\left(\frac{a}{2}\right)^2}.$$

On CA lay off $CD=CB\left(\text{or }\frac{a}{2}\right)$.

Then $$AD=\sqrt{a^2+\left(\frac{a}{2}\right)^2}-\frac{a}{2},\text{ or }x.$$

Therefore, on AB lay off $AF=AD$.



2. In a given equilateral triangle to inscribe another equilateral triangle whose area is equal to one half the given area.

Let ABC be the given equilateral triangle. If we lay off on the three sides the equal distances AD, BE, and CF, then $\triangle DFE$ is equilateral. To discover the length of AD, let $AB = a$, $AD = x$, and $FD = y$. Then the relation of the areas gives

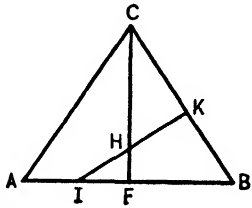
$$a^2 : y^2 = 2 : 1, \text{ or } y^2 = \frac{a^2}{2}. \qquad (1)$$

Since y in $\triangle ADF$ lies opposite an angle of 60°, we have

$$y^2 = x^2 + (a - x)^2 - x(a - x). \qquad (2)$$

$$\therefore \frac{a^2}{2} = 3x^2 - 3ax + a^2.$$

$$\therefore x = \frac{a}{2} \pm \frac{a}{6}\sqrt{3}.$$



To construct this expression draw the altitude CF. On FC lay off $FH = \frac{a}{6}$. Through H draw $IK \perp CB$, then $AI = x$.

CHAPTER XVI

IMPOSSIBLE CONSTRUCTIONS — REGULAR POLYGONS*

IMPOSSIBLE CONSTRUCTIONS

General principles. — Not *every* problem that may be proposed can be solved by means of ruler and compasses, and it is frequently of interest to determine whether or not a certain problem can be solved in this manner. While in some cases the answer to this question is exceedingly difficult, in others several general theorems make it a comparatively simple matter. Pure geometric investigations, however, cannot decide the matter; we have to employ the algebraic mode of attack that was discussed in the preceding paragraph.

Whether or not an algebraic expression resulting from such analysis can be constructed appears from the following proposition:

All rational expressions, and all expressions which contain square roots only (or can be reduced to such form), can be constructed. No others can be constructed.

Thus $a\sqrt[4]{2} = \sqrt{a(a\sqrt{2})}$ can be constructed.

But $\sqrt[3]{abc}$ cannot be constructed.

As the unknown quantity is always the root of an equation, it is desirable to determine what kind of equations have such roots, *i.e.*, roots that can be constructed.

* See F. Klein's Famous Problems of Elementary Geometry. Translated by Beman and Smith. Ginn and Co., N. Y., 1897.

*The roots of quadratic equations can always be constructed, and with one notable exception the roots of irreducible * equations of higher degree cannot be constructed.*

This exception embraces *certain* equations of degree 2^n which can be constructed and which will be discussed in the following section.

Thus, the roots of irreducible equations of the third or the seventh degree cannot be constructed. The application of these theorems will be shown in the next paragraph.

Three famous problems. — The three famous problems of antiquity that cannot be solved by means of ruler and compasses are:

1. *The duplication of the cube* (the so-called Delian problem).

2. *The trisection of an arbitrary angle.*

3. *The quadrature of the circle, i.e.*, the construction of a square equivalent to a circle (or the finding of π).

1. The Delian problem leads to the equation:

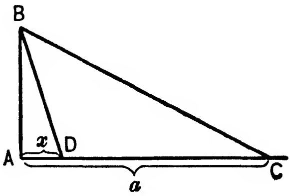
$$x^3 - 2 = 0.$$

This equation is irreducible † and not of the degree 2^n, hence its roots cannot be constructed, *i.e.*, the Delian problem cannot be solved by ruler and compasses.

* An equation $\phi(x) = 0$ is called irreducible if $\phi(x)$ cannot be resolved into rational factors.

† If this equation were reducible, it would have at least one factor of the first degree, and hence at least one rational root. But $x^3 - 2 = 0$ cannot have fractional roots, since the coefficient of x^3 is unity; it cannot have integral roots, since the factors of -2 do not satisfy the equation, *i.e.*, the equation cannot have rational roots.

2. The trisection of an arbitrary angle may be treated as follows: If B is the angle to be trisected, ABD or α is $\frac{1}{3}$ of B, $AB =$ unity, AC (or a) $\perp AB$, then we can construct $\angle \alpha$, if AD or x can be constructed.



Obviously $\tan B = a$,
and $\tan \alpha = x$.

But $$\tan B = \tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}.$$

Or $$a = \frac{3x - x^3}{1 - 3x^2}.$$

Hence $$x^3 - 3ax^2 - 3x + a = 0. \quad (1)$$

This equation is irreducible, for if the left member could be factored in its general form, it could be factored for any numerical value of a. But it cannot be factored for many values of a, e.g., $a = 2$.*

Hence the equation is irreducible, and since it is *not* of the degree 2^n, the problem cannot be solved by ruler and compasses.

3. The impossibility of squaring the circle cannot be demonstrated by the same simple method, since it is impossible to obtain any algebraic equation with

* If $a = 2$, the equation would be

$$x^3 - 6x^2 - 3x + 2 = 0.$$

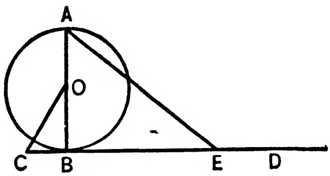
This equation cannot have fractional roots, since the coefficient of x^3 is unity; it cannot have integral roots, since no factor of 2 satisfies it. Hence the left member cannot contain a rational factor of the first degree.

rational coefficients whose root is π.* Hence no attempt will be made here to prove this case, and the reader is again referred to Klein's book.† Since π is a transcendental, and not simply an algebraic irrational number it cannot be constructed, even if we admit, besides circle and straight line, the use of other algebraic curves.

All three problems, however, can be effected by means of instruments other than straightedge and compasses.

The mechanical squaring of the circle by means of the so-called *integraph* has been accomplished rather recently by a Russian engineer, Abdant Abakanowicz.

Approximate constructions of π. — Approximate values of π, however, can be found by means of ruler and compasses only. The best known of these, which was given in 1685 by the Jesuit Kochanski, results in a value of π equal to 3.141533. That is, the resulting error is much smaller than average error due to unavoidable inaccuracy of drawing.‡ The construction is as follows: Draw diameter AB, and at B the tangent CD. Make $\angle BOC = 30°$, and $CE = 3\ (OB)$, then EA equals approximately the semicircumference.



* A number which does not satisfy any algebraic equations with rational coefficients is called a *transcendental* number.

† This proof for the impossibility of constructing π was first given by Lindeman in 1882. The demonstration was greatly simplified by Hilbert (Mathematische Annalen, Vol. 43).

‡ Still more exact is a construction given by G. Peirce. Bulletin of the American Math. Soc., January, 1907.

REGULAR POLYGONS

Division of a circumference. — The problem of dividing a circumference into n equal parts, *if n is prime*, was solved by the ancients for $n = 2$, 3, and 5; and no progress was made until in 1796 Gauss — then nineteen years of age — discovered the solution for $n = 17$, and demonstrated that the problem can be solved if n is of the form $2^k + 1$ and *prime*, and that it can be solved in no other case.

Gauss' discovery aroused the greatest interest among the mathematicians of his time, not only because he made an advance in a subject that had remained stationary for two thousand years, but also because he showed the connection between a geometric problem and an algebraic problem, viz., the equation $z^n = 1$.*

To make $2^k + 1$ prime, k must be a power of 2, *e.g.*, 2^t. Hence all prime numbers that represent constructible polygons must be of the form $2^{2^t} + 1 = n$. If $t = 5$, 6, 7, n is composite; if $t = 8$, or greater than 8, it is not known whether or not n is prime. Hence as far

* If z is a complex number, then the roots of the equation $z^n - 1 = 0$, represented in the customary graphical manner, determine the vertices of a regular polygon of n sides.

Since unity is a root of this equation, we may divide the left number by $z - 1$, and obtain:

$$z^{n-1} + z^{n-2} + z^{n-3} \ldots + z + 1 = 0.$$

Gauss showed that the roots of equations of this type can be constructed if $n - 1$ is a power of 2 and n is prime. In fact these equations represent the exceptional cases of higher equations with constructible roots, to which reference was made in the section on the possibility of solutions.

as known, all constructible polygons, the number of whose sides is prime, are represented by the cases $t = 0$, 1, 2, 3, 4. The corresponding number of sides of the polygons are: 3, 5, 17, 257, 65537.

Of this series only the first two have any practical value. The construction of the polygon of seventeen sides is already so complex that even with the most accurate methods of construction, the unavoidable inaccuracies in drawing make the result valueless for practical ends. The last two constructions have never been completed, although one mathematician devoted ten years of his life to the study of the last one.

In regard to composite values of n, Gauss showed that a polygon can be constructed if n is equal to the product of two or more different numbers of the series 3, 5, 17, 257, and 65537, but cannot be constructed for any power of these numbers. To divide a circle into eighty-five parts, we simply have to solve the indeterminate equation

$$\frac{1}{85} = \frac{x}{5} - \frac{y}{17}.$$

Since $x = 3$ and $y = 10$ are roots of this equation, we have $\frac{1}{85} = \frac{3}{5} - \frac{10}{17}$,
i.e., we many construct $\frac{1}{85}$ of the circumference by subtracting $\frac{10}{17}$ from $\frac{3}{5}$.

Similarly to get a polygon of $3 \times 5 \times 17$ or 255 sides, we solve the equation

$$\frac{1}{255} = \frac{x}{3} - \frac{y}{85}.$$

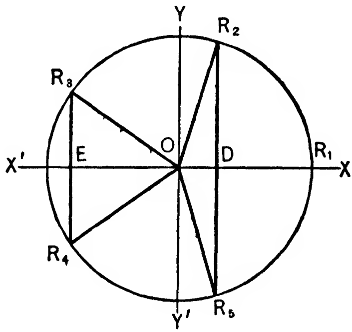
The solution gives

$$\frac{1}{255} = \frac{1}{3} - \frac{28}{85}, \text{ etc.}$$

Since an arc can be bisected, the construction can also be effected if the values of n obtained above are multiplied by any power of 2, as 2^m. Thus we obtain for $n \leqq 20$ constructible polygons if $n = 2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20$, while the regular polygon cannot be constructed if $n = 7, 9, 11, 13, 14, 18, 19$.

Concrete examples. — To show more concretely the connection between the construction of regular polygons and the solution of binomial equations, let us consider two problems.*

1. To inscribe a regular pentagon in the unit circle. Let XX' be the axis of real numbers, YY' the axis of imaginary, O the unit circle, and R_1, R_2, R_3, R_4, R_5 the vertices of the required pentagon.



Using the customary method for representing imaginary numbers graphically, it can easily be seen that $OR_1, OR_2 \cdots$ are the roots of the equation:

$$z^5 - 1 = 0. \qquad (1)$$

Since $OR_1 = 1$, we divide (1) by $z - 1$ and obtain

$$z^4 + z^3 + z^2 + z + 1 = 0. \qquad (2)$$

Equation (2) is a standard reciprocal equation, hence divide by z^2,

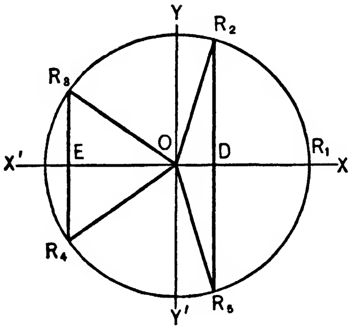
$$z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2} = 0,$$

* A knowledge of the graphic representation of imaginary numbers and kindred topics is necessary for these examples. See Advanced Algebra, p. 376.

and let
$$z + \frac{1}{z} = x.*$$

Thus we obtain after a few simplifications,
$$x^2 + x = 1.$$

Therefore
$$x = -\tfrac{1}{2} \pm \sqrt{1 + (\tfrac{1}{2})^2}.\dagger$$

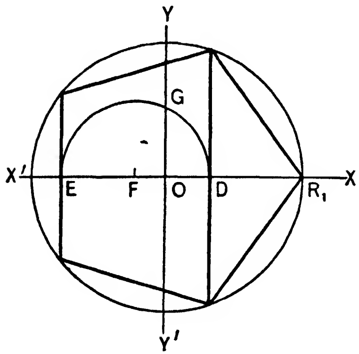


But x or $z + \frac{1}{z}$ can be interpreted geometrically.

If $z = OR_2$, $\frac{1}{z} = \frac{z^5}{z} = z^4 = OR_5$.

But graphic addition shows that the real part of OR_2, or OD, is equal to
$$\tfrac{1}{2}(OR_2 + OR_5),$$
or
$$OD = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{x}{2}.$$

Similarly it follows that if $z = OR_3$, then $OE = \frac{x}{2}$.



Therefore the two values of $\frac{x}{2}$ represent the real parts (the abscissas) of the four required points.

But
$$\frac{x}{2} = \pm\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2} - \frac{1}{4}.$$

Hence we have the following construction:

On OX' lay off $OF = \tfrac{1}{4}$. On OY lay off $OG = \tfrac{1}{2}$. From F with radius FG draw a circle, meeting XX' in E and D. The perpendiculars erected at E and D determine the vertices of the required pentagon.

* Advanced Algebra, p. 498.

† For geometric constructions, expressions such as the above should not be simplified further.

2. To construct a regular polygon of seven sides.

In the same manner as in the preceding example we obtain

$$z^7 - 1 = 0. \qquad (1)$$

$$z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0.$$

$$z^3 + z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} = 0. \qquad (2)$$

Substituting $x = z + \frac{1}{z}$, we arrive at the cubic

$$x^3 + x^2 - 2x - 1 = 0. \qquad (3)$$

If this equation were reducible, it would have at least one factor of the first degree, and hence at least one rational root. But (3) cannot have fractional roots, since the coefficient of x^3 is unity; it cannot have integral roots, since neither $+1$ nor -1 satisfy it. Hence (3) is irreducible, and its roots and therefore the roots of (1) cannot be constructed with ruler and compasses.

It is therefore *impossible* to construct a regular heptagon with ruler and compasses.

Constructible angles. — The discussions of the preceding sections also answer the question as to the angles which can be constructed by ruler and compasses only. Every central angle corresponding to one of the above polygons can be constructed, and of course their sums and differences. Thus $n = 9$ produces an angle $\alpha = 40°$, $n = 15$, an angle $\alpha = 24°$, etc. It is sometimes useful to know that the smallest constructible angle that can be expressed in an integral number of degrees is 3°, and that hence an angle which contains degrees only is constructible if it is a multiple of 3, and not constructible if it is not a multiple of 3. Thus 27°, 39°, 54°, and 87° can be constructed. 11°, 25°, 37°, cannot be constructed. The angle of 3° is easily obtained by repeated bisection of 24°.

CHAPTER XVII

SOME REMARKS ON SOLID GEOMETRY

PURPOSE AND DIFFICULTIES OF THE STUDY

Peculiarities of solid geometry. — Under the conditions that prevail in our secondary schools, and with the time usually allowed for the subject, it seems that solid geometry cannot be made a subject of discovery and a discipline for training the mind to quite the same extent as plane geometry. Exercises in solid geometry of purely demonstrative character are comparatively hard to construct and often more difficult to solve than those in plane geometry; the amount of book matter to be studied is relatively large, and the danger of making students learn by heart many proofs which they would never discover themselves, is greater than in plane geometry.

On the other hand, numerical examples of great simplicity can be given almost throughout the course. It is easier to find practical applications of the theory, and algebraic work that gives to the student practice in the various uses of a formula can be frequently introduced. Moreover, the study of solid geometry strengthens the student's space imagination and his power to image space configurations, and it gives him an understanding for drawings that represent spatial objects.

Altogether it seems that the utilitarian advantages are somewhat greater, but the purely cultural advantages somewhat smaller, than in plane geometry. In accordance with this view, textbooks devote a large part of their space, and schools a large part of their time, to mensuration and to theorems that, directly or indirectly, lead to mensuration. Such an arrangement seems to be perfectly justified under present conditions, and in many cases it may be recommended to cut down the number of theorems even further, to omit the difficult ones that are not absolutely needed, as, *e.g.*, the one about the common perpendicular of two lines, and not to insist upon a knowledge of the proofs of the most difficult ones that cannot be dispensed with, as, *e.g.*, the theorem of the equivalence of triangular pyramids.*

Difficulties. — With such restrictions the study of solid geometry will not offer great difficulty to the student. It may require a little more time and a little more study, but it does not require more intelligence than does plane geometry.

One difficulty, however, against which we must guard and which we must overcome at the very start is the inability of some students to understand diagrams of solids. There are students who are able to reason logically, but who cannot imagine clearly the spatial forms

* Various attempts have been made to demonstrate this theorem by dividing the two pyramids into parts which are respectively congruent and thus to avoid limits. Recently, however, it was proved that this is impossible.

which the diagrams represent. There are two ways of overcoming this difficulty, namely, the use of models and rational methods of drawing.

MODELS

The function of the model is to help the student in the beginning to an understanding of solid figures in general, and to make clear to him, later on, difficult drawings which otherwise he would not understand. The model should, however, not be used to supplant the drawing. As soon as the student is able to understand the drawings, the models should be discarded or reserved for the most difficult cases only. Otherwise the student will lose one of the main benefits of the study, viz., the development of his space imagination and of his faculty to understand diagrams of solids. Matters difficult to depict, however, such as the regular polyedrons, the distinction between right and rectangular parallelopiped, etc., should be explained by the use of models.*

Kind of models needed. — As a rule the simple, inexpensive model made of paper, strings, wire, etc., serves its purpose just as well as the most expensive one. A triangular prism, cut out of a potato, and divided into three equivalent pyramids is just as instructive as the most expensive model. Many of the propositions relat-

* Another reason why models should be used only sparingly is that, as a rule, they cannot be shown to a large class as readily as a blackboard diagram, but must be explained to the students individually or to small groups of students, thus sacrificing a great deal of time.

ing to lines and planes may be illustrated by a couple of pencils, a book, a piece of paper.

The only expensive piece that is quite useful is a spherical blackboard, although even here some substitute, *e.g.*, a football, may be found.

As far as the making of models by students is concerned, there can be no doubt that in many cases the student's understanding will be improved thereby. A student who makes a regular icosaedron, or dodecaedron out of cardboard has undoubtedly a much clearer notion of these solids than he had before.

On the other hand, it is doubtful whether the expenditure of a great deal of the student's time for the making of many or elaborate models is justified. The time may be well invested as far as manual training is concerned, but not as far as it relates to mathematical reasoning. Especially the custom of exhibiting mathematical work and models made by students must be strongly condemned. Such exhibitions not only raise an utterly wrong standard for the measurement of the result of mathematical work, but they are often like many educational exhibits — deceptive and misleading.

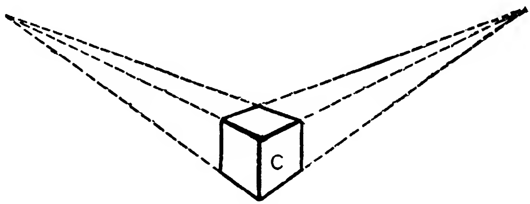
DRAWINGS

Photos or drawings? — Some textbooks are equipped with photos of models, which by their clearness almost equal models. But to the continuous use of such photographs the same objections may be raised as to the continuous use of models. Hence shaded diagrams, which can be made very perspicuous and which the student

can redraw, are as a rule preferable. Photographs, however, will be very serviceable if used for a few exceedingly difficult cases only.

Perspective or projection? — It is impossible to give within the limits of this textbook a full explanation of perspective and projection.*

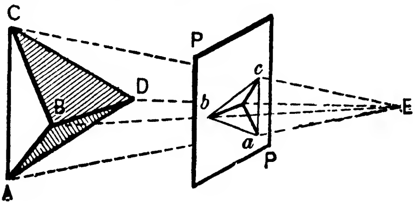
Perspective produces a picture as the eye sees the object, or as a photo depicts it, as illustrated by the



annexed perspective C of a cube. All vertical edges appear as vertical lines; all other parallel edges converge towards a point.

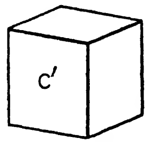
An orthographic projection of an object is a picture

* We obtain an image of the point A upon the plane PP as viewed from point E, by joining E (the eye of the observer) to A. Then a, the intersection of AE and PP, is the required image. Thus $abcd$ is an image of the pyramid of $ABCD$ upon the plane PP.



If the eye (E) is at a finite distance, the drawing is a *perspective;* if E is at an infinite (or very large) distance, so that the rays EA, EB, EC, etc., are all parallel, it is called a *projection*. According as the rays EA, EB, etc., are perpendicular or oblique to PP, we obtain either an *orthographic* or an *oblique* projection.

as the eye would see it if it were very far off, or if the object were comparatively small (C'). Parallel edges have parallel projections, and the length of parallel lines is proportional to their projections. Hence projection allows a measurement of the three dimensions of a solid, a problem that is very difficult for perspective.



While a superficial view of this matter may lead to preference of perspective for the drawings of solid geometry, the consensus of opinion of all who have given serious thought to the subject is in favor of projections, principally on account of the following reasons:

1. The alleged superiority of perspective views — being views as the eye sees the object — is imaginary, as the true views of small models seen by a student from a distance of about 10 feet differ so little from projections that the eye would not notice the difference.

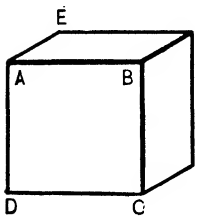
2. Perspective views are far more difficult to draw than projections, and hence students as a rule draw exceedingly inferior pictures if they use perspective. Usually they exaggerate the convergence of parallel lines. If students really constructed their diagrams in solid geometry, this would not happen, but there is no time to study such constructions. The diagrams are simply drawn according to the artistic intuition of the student and according to his ability to redraw pictures that he has seen.

3. Lines that are parallel have parallel projections. This is a great help for drawing, and also avoids confusion in proofs.

4. The length of parallel lines is proportional to their projections.

5. Projections are used almost exclusively for technical and engineering purposes.

Oblique projection. — To construct an oblique projection of a cube whose edge equals one inch, draw the square *ABCD*, whose edge is one inch, and draw all edges which are perpendicular to face *ABCD*, as *AE*, equal to a certain assumed fraction ϵ ($=\frac{1}{2}$ or $\frac{2}{3}$) of one inch, and inclined a certain assumed angle α (about 30°) to the horizontal line *AB*.*



Oblique projections are easier to draw than orthographic projections, but give badly distorted views, while the orthographic projections (see diagrams on pp. 274 and 276) are approximately views as the eye sees them.

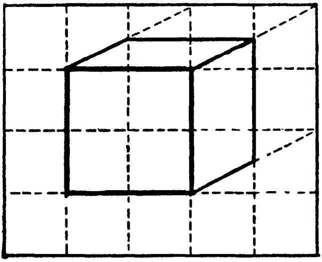
For very complex drawings, oblique projection is to be preferred; for simple drawings, however, such as a plane, a cube, a sphere,† orthographic projection is surely better. One should attempt to carry on orthographic projection — the drawing method of the engineer — as far as possible, and textbooks should give, as a rule, orthographic projections.

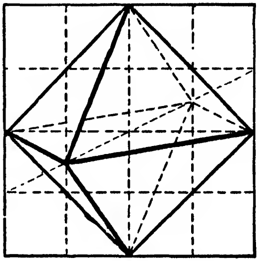
Use of cross-section paper. — The drawing of oblique projections is greatly facilitated by the use of ordinary

* The shadow of a wire frame of a cube resting with one face upon the paper is an oblique projection.

† The orthographic projection of a sphere is a circle, while the oblique projection and the perspective are ellipses.

rectangular cross-section paper, as illustrated by the annexed diagrams of a cube and an octaedron. (Here we assume $\alpha = \tan^{-1}\frac{1}{2}$, $\epsilon = \frac{1}{4}\sqrt{5}$.)

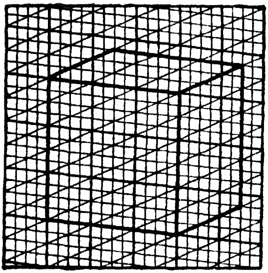


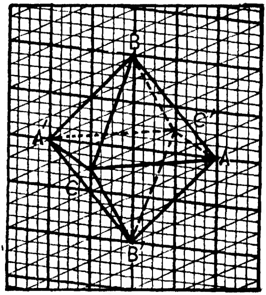


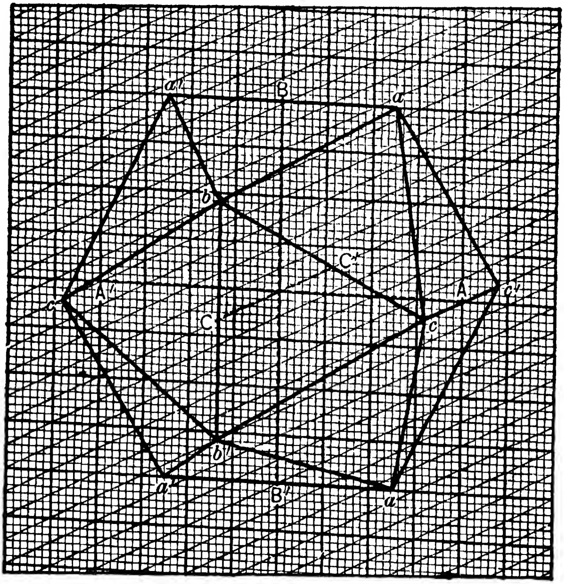
To simplify the exact construction of orthographic projection, the author has devised a cross-section paper, called *trimetric* paper, that enables the student to draw at once the orthographic projection of any line of known length, that is parallel to one of the three principal axes, and with little construction the projection of any line, straight or curved. Consequently an orthographic projection of any solid is easily obtained.*

All diagrams of solids which are contained in this chapter are constructed by means of this trimetric paper. The first of the three diagrams on the next page represents a cube whose edge equals $\frac{3}{4}$ of an inch ($\frac{1}{4}$ is the unit of the paper). This cube also explains to a certain extent the use of the paper, in particular the lengths and directions of the three principal axes.

* For details the reader is referred to the author's article on trimetric paper in the *Engineering News*, New York, April 6, 1911.



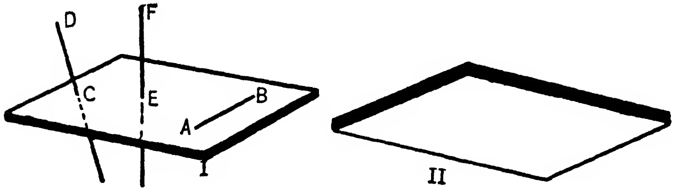




The second drawing represents a regular octaedron whose axes equal one ($\frac{4}{4}$) inch. (Draw the three axes *AA'*, *BB'*, *CC'* equal to four units and bisecting each other.)

The third diagram is the projection of a regular icosaedron whose axes, AA', BB', and CC', equal two inches. The edges aa', bb', and cc' are parallel to the corresponding axes, and equal to $\frac{5}{8}$ * of these axes. †

Minor rules for drawing.— 1. In representing a plane draw a material plane that has thickness, as I and II. While the two projections have equal outlines, I is seen from above, II from below.



2. Lines *in* a plane are represented by projections which terminate within the boundaries of the plane, as AB. The projections of lines which intersect the plane

* $\frac{5}{8}$ is an approximation of the ratio which the greater part of a line that is divided in extreme and mean ratio bears to the entire line. These approximations are convergents of the continued fraction:

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1\ldots}}}$$

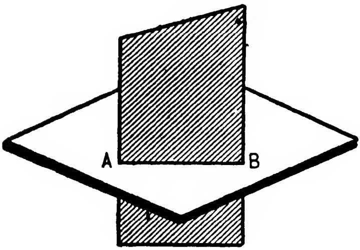
For most purposes $\frac{3}{5}$ (error about 2 %) is sufficient. More exact are $\frac{5}{8}$ and $\frac{8}{13}$.

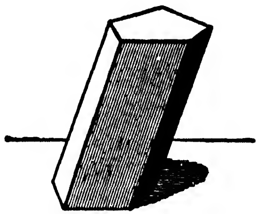
† This outline of the construction of the regular icosaedron is intended only for students who are familiar with the general principles of constructing projection of regular solids. For details of such constructions see the excellent German book on Solid Geometry: Holzmüller, Stereometrie, Band I, Leipzig, 1900. This book is a regular treasury of information in regard to many of the most interesting questions of Solid Geometry.

extend beyond the boundaries of the plane, as *CD*. Lines perpendicular to plane I are perpendicular to the lower edge of the paper, as *EF*.

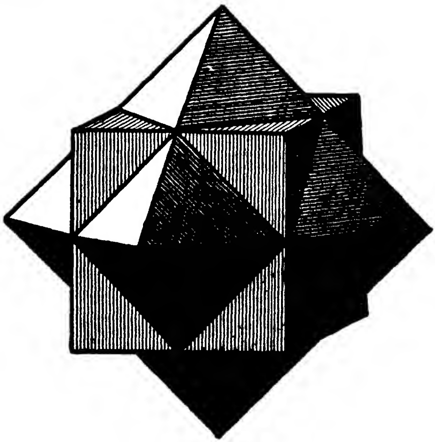
3. Lines that cannot be seen omit or draw dotted. (See No. 1 and No. 4.)

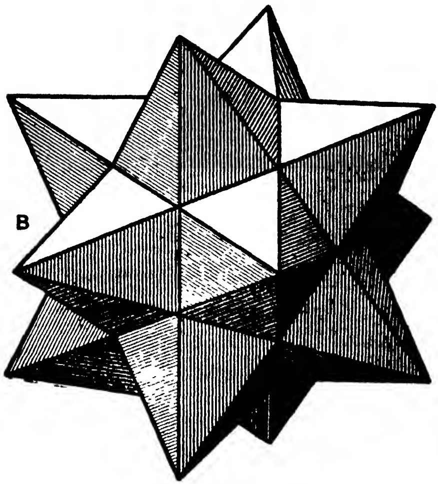
4. Intersecting planes sometimes appear clearer if they lie so that their line of intersection, *AB*, does not extend to the margin of one of the planes.





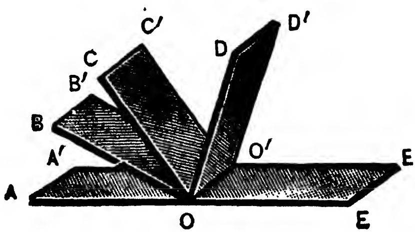
5. Lines near to the eye draw heavier than the remoter ones.



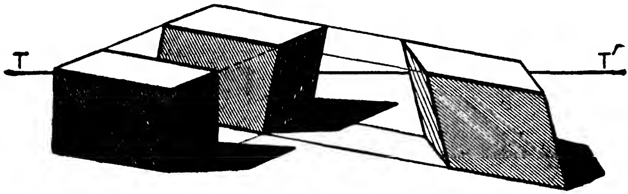


6. Point out the difference in the position of different planes by shading (the light is usually assumed to come from the left side).

7. To draw complex figures containing several diedral angles draw first the corresponding plane figure and then make each side a plane. Thus to draw two adjacent supplementary diedral angles and their bisectors draw first the corresponding plane figure *OABCDE*. Then draw an arbitrary (short) line *OO′* and make *AA′*, *BB′*, *CC′*, etc., equal and parallel to *OO′*.



8. Some drawings become clearer by the introduction of the surface of a table upon which the solid rests, and the shadow cast upon the table.



In the annexed diagram TT' is supposed to be the table line.

CHAPTER XVIII

APPLIED PROBLEMS

Practical and scientific application of elementary mathematics. — The greater part of all that is written and uttered to-day on the subject of the reform of mathematical teaching is connected with the introduction of applied mathematics into our secondary school courses. It is claimed that real problems relating to daily life or to technical and scientific matters will redeem mathematical study in secondary schools from its inefficiency. The failure of the mathematical instruction to arouse and to hold the interest of the students, the absolute forgetting of all that has been studied, and the inability of pupils to apply what is known, are said to be due, solely or principally, to the fact that the pupil does not see any practical value in either algebra or geometry.

While it is somewhat unfortunate that nearly all reformers concentrate their efforts upon this one point, to the neglect of other more important points, it cannot be denied that this tendency in itself is a good one, provided it is confined to its proper limits. Mathematics owes its very existence to its applications, and it would never be taught in a school if it were a mere collection of symbols that are logically coherent, but with no relation whatsoever to the real world. There are, how-

ever, different degrees to which these ideas may be carried, and the extreme ends of this scale of views are so far apart that the practical results will be utterly different according as we accept the moderate or the extreme view of this movement.

Moderate views. — Elementary mathematics has not very many genuine applications, but still it has some. The study of these applications will undoubtedly increase the interest in the subject; frequently it will also lead to a better understanding of the subject; and occasionally it may be of practical value to some student. On the other hand, we must not forget that the principal value of mathematical study lies in the mental training it affords, and hence we should not give up any of the essentials of the subject because they have no immediate applications. We must not, in order to obtain practical value, deprive the subject of its peculiar character of being a subject of reasoning. But *other things being equal,* that topic deserves preference that can be applied or that will *ultimately lead* to applications. *Other things being equal,* the practical problem deserves precedence over the purely academic problem.

Thus we shall not give up the theory of exponents, or the analysis of geometric problems, simply because they have no immediate practical value. But when we have to choose between several topics, each of which is not absolutely essential for the whole theory, we should be guided by their applicability and should not merely follow tradition. Thus we may very well dispense with some of the complex "cases" of factoring, and study

instead graphical methods, which have a greater practical value than any other chapter of elementary mathematics. We may give up the finding of the H. C. F. by the Euclidean method, and introduce instead proportion, which is necessary in physics and geometry. Instead of going very deeply into goniometry, it would be better to devote our time to the trigonometric methods for finding heights and distances. Instead of substituting numbers in expressions formed at random, let us study numerical substitution in formulæ of practical value, etc., etc.

The extreme view sees the chief value of mathematical instruction in its utility, and considers its disciplinary value as quite small. The present inefficiency of mathematical instruction is said to be due to the very nature of pure mathematics, which cannot be understood fully by young students, and which cannot possibly interest them. Only topics the practical value of which is apparent are said to interest young people, and hence it is proposed to make the applied problem the principal, if not the only, object of instruction. Purely academic problems and theorems are to be admitted only as far as they are absolutely necessary for the solution of practical problems.

The only criterion by which any particular problem or theory is judged is its applicability. Thus the problem, "To construct a triangle, having given the perimeter, one angle, and the altitude from the vertex of the given angle," is considered poor, because of its "having at best remote connection with any uses of geometry

within reach of the ordinary high school pupil." * No other reason for using the above problem is therefore admitted; there is only one, viz., its applicability.

Reasons against making mathematics a utilitarian subject. — Obviously the matter of applying problems can be carried too far, and among the many reasons that may be given against the extreme views stated in the preceding paragraph, the following may be mentioned:

1. The assumption that students cannot successfully study pure mathematics, and never take any interest in purely academic problems and theorems, is erroneous. Students of average ability, properly prepared, taught by proper methods under favorable conditions, not only understand mathematics easily, but take great interest in the subject. That the conditions in many schools are such that good teaching is almost impossible, should make us attempt to change these conditions, but not reject one of the best and most interesting subjects.

2. If all secondary school mathematics had genuine applications, the proposed changes would not so much interfere with the teaching of the essentials of geometry and algebra. But the field of application of these subjects is limited, and a great many of the so-called applications are not genuine applications. To replace in a time-honored algebraic problem the number of Henry's marbles by the height of Chimborazo, or A's age by the number of babies born annually in Chicago, does not lead to a genuine application of algebra, since nobody

* Provisional Report of the National Committee of Fifteen on Geometry Syllabus, *School Science and Mathematics*, May, 1911.

would find the height of Chimborazo or the number of babies born in Chicago by such a method.*

Genuine applications of algebra may often be taken from physics, but unfortunately the average pupil's knowledge of physics is so small that an extensive use of such problems involves as a rule the teaching of physics by the teacher of algebra.

Still more restricted is the field of true application in geometry. Examples relating to Gothic windows and parquet floors are sometimes interesting, but they relate only to a very small fraction of the geometry, and they are rarely genuine, since the draftsman, the glazier, or the carpenter who has to deal with these forms will as a rule solve all problems involved empirically.

Thus, while the applied problem does increase the interest, it is too limited in scope to be made the fundamental principle of the teaching of mathematics.

3. While a certain proportion of applied work will stimulate the interest, an accumulation of a large number of applied theorems of a similar kind proves tiresome in the end. To get a few statistical facts is interesting; to hear of endless numbers of such facts is like reading the World's statistical almanac. A few examples relating to parquet floors are attractive, but pages of them void of interest.

4. To put the utility of mathematics above its culture value is decidedly based upon a misconception of the educational value of the subject.

* In addition, on account of the large numbers involved, arithmetical difficulties appear that are utterly foreign to our purpose.

5. To make the applicability of a problem or other topic the only criterion by which to judge it means utterly to ignore other reasons that exist for introducing such matters. A problem is given to illustrate a new idea, a new method, and that problem which best illustrates these methods is the best. If we wish to illustrate the analysis of problems, we may very well use the problem of constructing a triangle having given the perimeter, one angle, and one altitude, to which reference was made in the preceding section. It is certainly better than a problem that does not teach analysis, even if it should relate to "baby ribbons," "6-inch bias ruffles,"* and other "practical matters."

6. If we make mathematics a practical subject with little regard to its disciplinary value, it will not only lose a great deal of its beauty and dignity, but it will be very difficult to defend the teaching of the subject against the numerous attacks of its enemies, and its early disappearance from the curricula of our schools will not be unlikely.

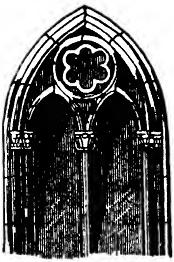
Conclusion. — Practical problems, introduced rationally and without destroying the essentials of geometry and algebra, will improve the teaching of mathematics. But we must not expect too much from this movement, for it is not a panacea that will cure all ills. There is even a danger that a fanatical pushing of this idea may do serious harm to the cause.

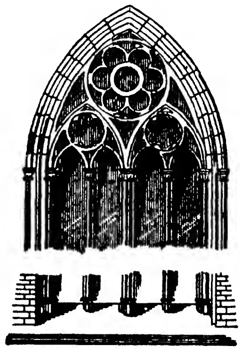
Sources of applied problems. — In many textbooks and magazine articles there may be found a large number

* These terms are found in some recent textbooks.

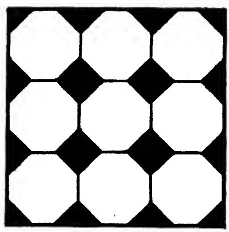
of practical problems, relating to engineering, physics, commercial life, etc. Bibliographies relating to such sources are found in *School Science and Mathematics*, Vol. VIII and Vol. XI.* Some of the most useful and most numerous examples belong to the following types:

1. *Geometric constructions.* — Most applied problems relate to architectural forms and ornamental designs such as found in parquet floors, linoleums, Gothic windows, etc. Thus the first of the annexed diagrams





requires the drawing of three equilateral gothic arches, and the construction of a circle touching these arches; the second diagram contains several similar problems.



Similarly it may be required to *construct* the annexed linoleum pattern so that the black parts may be squares, and the white parts regular octagons, etc.

* Vol. VIII, No. 8, November, 1908, pp. 641–644; Vol. XI, No. 5, May, 1911, pp. 458–460 (Report of Fifteen).

Maps may also be used to applied constructions. We may require the construction of the site of a schoolhouse that is equidistant from three towns A, B, and C, or the location of a railway station to be equidistant from two towns A and B.

Problems of subdividing areas, such as fields, belong also to this class of applied problems.

2. *Geometric computations.* — The forms considered in the preceding class give rise to many numerical problems, *e.g.*, the finding of the edge of the octagon in the linoleum pattern, the calculation of the radius and circumference of the circle in the gothic window. A large number of exercises can be based upon the calculation of areas and volumes of forms met in engineering and architecture, as pipes, boilers, tanks, arches, etc.

Problems relating to heights and distances similar to those given in trigonometry can be solved if the given angles are 30°, 45°, 60°, 90°, 120°, 135°, and 150°. (Theoretically if they are multiples of 3° and their halves, quarters, etc.) Thus the simplest mode of finding a distance would be by means of an equilateral triangle obtained by measuring two angles equal to 60°. Then we may consider right triangles, containing one angle, equal to 30°, or 45°, etc. We may introduce angles of elevation and depression, angular distances, nautical notions, precisely as in trigonometry, although the more complex examples of this kind will prove too difficult for the average student.

Thus we may ask for the height of a tower standing on a horizontal plane, if at a certain point on the plane

the angle of elevation of the top of the tower is 30°, and at a point 80 feet nearer the angle is 45°.

3. *Algebraic problems* are in general better known, and found in most textbooks of algebra. Physical and commercial problems form one source, the old-fashioned marble problems clothed with a new garb by statistical data furnish another. Graphic methods are particularly fertile in producing applied problems.*

* See also Chap. XIX, p. 299, and Chap. XX, p. 335.

CHAPTER XIX

THE CURRICULUM IN ALGEBRA

INTRODUCTORY REMARKS

The educational value of algebra as compared with geometry. — The selection of the subject matter for courses in elementary algebra must largely depend upon the educational advantages of the subject, which are not absolutely identical with those of geometry. Algebra requires the same accuracy of thinking, and the same, or possibly greater, accuracy of detail than geometry. It may be graded as perfectly, and its introductory chapters may be made even simpler than those of geometry. The definiteness of the task given to the student, the certainty of the results, and the applicability of many of its topics to scientific or other problems are precisely the same as in geometry.

On the other hand, algebra does not require *as much* reasoning, and this reasoning is not always of the same high order as geometry.* The amount of information cannot be reduced quite as much as in geometry, and some topics in algebra require a certain amount of mechanical drill. Hence ingenuity and originality of thinking do not play quite the same rôle, and the knowledge of facts is somewhat more important than

* Compare Chap. II, p. 23.

in geometry. Moreover, algebra lends itself rather readily to a purely mechanical treatment. Students may add exponents, transpose terms, and perform other manipulations without having a clear notion of the meaning of these operations, and the symbols involved.

Thus, while possessing most of the advantages of other mathematical branches, algebra has certain drawbacks, and the courses of study should be so arranged as to eliminate or to minimize these disadvantages.

When should algebra be studied? — In most schools in the United States algebra is studied before geometry,* but lately this plan has been frequently assailed; sometimes it has even been considered one of the chief causes of the inefficiency of mathematical teaching. While some reformers wish to place geometry before algebra, most of them advocate a simultaneous teaching of the two subjects.†

The feasibility of the first plan under certain conditions is proved by the experience of a number of schools in various countries. In regard to the second plan, it should be emphasized that it does not mean the teaching of 3 hours of algebra and 2 hours of geometry during the first two years of high school, but a complete merging of the two subjects. Any new method, whether it

* Usually algebra during the ninth, geometry during the tenth, school year.

† The three most widely advocated reforms of mathematical study are:

1. The Use of Laboratory Methods.
2. The Teaching of Applied Problems.
3. Simultaneous Teaching of Algebra and Geometry.

relates to geometry or algebra, should, according to this plan, be studied whenever the necessity for it arises, and not before. Thus, square roots should be taught in connection with the Pythagorean theorem, similar triangles with proportion, etc. All algebraic facts should as far as possible be illustrated geometrically, and *vice versa.*

This scheme, which has been advocated not only in the United States, but also in other countries, especially in Italy and Germany, has undoubtedly a number of advantages. It may arouse more interest than the customary mode, it may at some stages of the work show the student the necessity of studying certain topics, it may train the student better to apply his knowledge, and it may prevent the rapid forgetting of algebra during the time when geometry alone is studied.

On the other hand, there are weighty reasons against the introduction of simultaneous teaching of algebra and geometry in every school. First of all, such a complete merging of the two subjects may not be at all possible. The courses of study of this new type, which have been proposed, are still in the experimental stage, and, as a rule, lack detail. The textbooks that pretend to carry out this idea, usually do the merging for one or two topics, otherwise they simply alternate between algebraic and geometric topics.* Moreover, there exist

* Of course, there can be no difference of opinion about the wisdom of introducing algebraic illustrations into geometry, and *vice versa,* whenever feasible. But there is a wide difference between such a procedure and complete merging.

a large number of high schools in which the first year students are not capable of attacking geometry as successfully as algebra. Trained principally in mechanical modes of study, such students find the transition from arithmetic to geometry too difficult, while they attack algebra — which resembles arithmetic — quite successfully.

Hence it seems doubtful whether the simultaneous teaching of algebra and geometry would produce such a radical improvement as the advocates of this plan claim. Still the plan is worth trying, and schools that are in a position to make experiments should give this matter a thorough and impartial trial.

WHAT ALGEBRA SHOULD BE STUDIED

General remarks. — There can be no discussion about the adoption of a large number of topics in courses in elementary algebra; namely, of all those which are absolutely needed for further work. There are, however, a number of other topics that may be dispensed with, and, in regard to these, until a few years ago there was no accepted criterion for introduction other than tradition.*

Lately, however, there is noticeable an almost universal tendency to eliminate traditional subjects and to put in their place those having pedagogical or

* In some of these cases it was claimed that future applications depended upon them, but these applications were as superfluous as the original methods. Thus the teaching of certain complex cases in factoring was defended because the solution of certain equations depended upon them, but a closer inquiry showed that the applications were as superfluous as the factoring method.

practical value. The general tendency of these proposed changes may be briefly characterized as follows:

1. Reduction of all information to a minimum: elimination of all superfluous, abstract, and merely technical matter.

2. Emphasis on all algebraic topics that require original thinking.

3. Emphasis on all topics that are frequently applied in geometry, physics, engineering, commercial problems, etc.

Reduction of the amount of purely formal work. — Algebra cannot be mastered without the study of a number of formal operations and without acquiring a certain amount of manipulative skill. In order to attack successfully the more advanced chapters of algebra, the student has to be familiar with the four fundamental operations; he has to possess a certain skill in factoring, in solving equations, etc. But, on the other hand, these matters can be, and have been, carried too far. Until recently there was a tendency to overemphasize this manipulative phase of the work, and to increase it from year to year. The authors of textbooks vied with each other in their attempts to put everything into their books that possibly could be taught.*

The cause for this tendency was a twofold one. In the first place, algebra was taught largely for the

* Some authors stated with great self-satisfaction in their prefaces that they had increased the number of cases, *e.g.*, added a fourteenth case of factoring to the thirteen commonly taught.

sake of preparing students for examinations, and hence it was attempted to make, as far as possible, every example a special case of a memorized method, thus reducing the study almost to a mechanical application of memorized rules. Second, it was claimed that the study of such formal methods, as, for instance, the extraction of the cube root of a polynomial, possessed great disciplinary value, and hence that the acquirement of manipulative skill should be the aim, or at least one of the principal aims, of algebraic study.

Any one, however, who is familiar with our schools knows that these formal matters are studied in a way that makes their disciplinary value small. Thus, even if the teacher should at first emphasize the reasoning that underlies the extraction of a cube root of a polynomial, very soon the students will perform this manipulation in a purely mechanical manner. They simply know the process by heart, and are utterly unable to reconstruct the same, if they should forget it. Moreover, there is so much purely formal work in algebra that its reduction by one third or one half will not sensibly diminish the educational benefits that may be derived from it.

Fortunately the mathematical public commences to recognize the uselessness of overemphasizing the study of mere manipulative work. In the United States,* in

* The requirements of the College Entrance Examination Board — which are based upon a committee report of the American Mathematical Society of the year 1902 — eliminate several of the time-honored methods. Later reports, essays, and textbooks show a marked tendency in the same direction.

France, in Germany, in England, there are marked tendencies to reduce the purely technical side of algebraic study.

It is impossible to make rules in this respect that fit every kind of school, but among the topics that *may* be omitted — and in many cases *should* be omitted — may be mentioned:

1. * Complex cases of the removal of parentheses. (One parenthesis within another is sufficient.)

2. Multiplication and division of powers with literal exponents (when this topic is first studied).

3. Complex cases of multiplication and division of polynomials by polynomials.

4. The Euclidean method for finding the H. C. F. In elementary algebra this method is needed only for the reduction of a fraction to lowest terms. But no practical example leads to fractions whose terms are of the third or fourth degree.

5. The analogous method for L. C. M. is needed only for the addition of fractions, but fractions whose denominators are cubic functions have no value in elementary algebra.

6. The addition of fractions with quadratic or cubic denominators.

7. * The greater part of factoring.

8. Complicated complex fractions.

9. The method of comparison for solving simultaneous linear equations.

* Topics marked by an asterisk (*) will be discussed in detail in Chapter XX.

10. Cubes and higher powers of polynomials.

11. Cube roots of polynomials.

12. Cube roots of numbers.

13. Difficult simultaneous quadratics, especially those solved by devices which the student cannot discover himself (*e.g.*, symmetric equations).

14. Nearly all so-called short cuts and special devices.

15. The greater part of the theory of algebra.

In the more advanced chapters we may omit:

16. Recurring series.

17. The greater part of inequalities.

18. Sturm's Theorem.

19. Multiple roots.

The reduction of the formal work in algebra must not, however, be interpreted as involving a less thorough study of the topics that are retained. Rather the contrary. The fundamental algebraic operations should be studied thoroughly and should be made familiar to the student by frequent repetition. Mathematics is a simple study for one who never attacks a new topic until he is thoroughly familiar with the preceding stages of the work, while lack of this familiarity makes further progress very difficult, if not impossible.

Reduction of theory. — All parts of the theory which are beyond the comprehension of the student or which are logically unsound should be omitted. Every practical teacher knows how few students understand and appreciate the more difficult parts of the theory, as the proofs of the fundamental laws relating to negative or fractional exponents, etc. Even matters

as simple as the multiplication and division of fractions $\left(\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}, \text{ and } \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}\right)$ are usually not fully comprehended by the pupils.

Moreover, some of the proofs offered in the textbooks are logically unsound, as the "proof" for the law of signs in multiplication, and the proofs of the binomial theorem for negative or fractional exponents.

Elimination of short cuts and special devices. — One of the most interesting problems for the exceptionally able student is the discovery of simple and short solutions of problems which, treated by the general methods, lead to lengthy or awkward solutions. It would be very difficult to solve by the regular method the equation

$$\frac{x-1}{x-2} - \frac{x-2}{x-3} = \frac{x-4}{x-5} - \frac{x-5}{x-6}.$$

A series of special devices, however, can effect a very simple solution.

If we represent each fraction as a mixed number,

e.g., $$\frac{x-1}{x-2} = 1 + \frac{1}{x-2},$$

the equation is easily reduced to

$$\frac{1}{x-2} - \frac{1}{x-3} = \frac{1}{x-5} - \frac{1}{x-6}.$$

Instead of clearing the equations of fractions, we simplify each member. Thus we obtain

$$\frac{-1}{x^2-5x+6} = \frac{-1}{x^2-11x+30}.$$

Or $$x^2-5x+6 = x^2-11x+30.$$

Therefore $$x = 4.$$

The preceding method is an example of a very simple and obvious device, but many of those used in elementary algebra are far more complex, and their use is justified only by the result. Of such a kind are the substitutions used sometimes in the solution of simultaneous equations; as the substitution:

$$x = vy, \text{ or } x = u + v,\ y = u - v, \text{ etc.}$$

A good example of a short cut is the following transformation of a radical into an infinite continued fraction. The regular method applied to the finding of $\sqrt{22}$ is as follows:

$$\sqrt{22} = 4 + \frac{\sqrt{22} - 4}{1},$$

$$a = \frac{1}{\sqrt{22} - 4} = \frac{\sqrt{22} + 4}{6} = 1 + \frac{\sqrt{22} - 2}{6},$$

$$b = \frac{6}{\sqrt{22} - 2} = \frac{\sqrt{22} + 2}{3} = 2 + \frac{\sqrt{22} - 4}{3},$$

$$c = \frac{3}{\sqrt{22} - 4} = \frac{\sqrt{22} + 4}{2} = 4 + \frac{\sqrt{22} - 4}{2},$$

$$d = \frac{2}{\sqrt{22} - 4} = \frac{\sqrt{22} + 4}{3} = 2 + \frac{\sqrt{22} - 2}{3},$$

$$e = \frac{3}{\sqrt{22} - 2} = \frac{\sqrt{22} + 2}{6} = 1 + \frac{\sqrt{22} - 4}{6},$$

$$f = \frac{6}{\sqrt{22} - 4} = \frac{\sqrt{22} + 4}{1} = 8 + \frac{\sqrt{22} - 4}{1},$$

$$\sqrt{22} = 4 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{8 + \cfrac{1}{1 + \text{etc.}}}}}}}}$$

The short cut, however, gives the integral number (first value of A) and the successive denominators (C) as follows:

A	B	C
4	6	1
2	3	2
4	2	4
4	3	2
2	6	1
4	1	8

The first value of A is the largest integer contained in $\sqrt{22}$, *i.e.*, 4. The first value of B equals $22 - A^2 = 6$. Any value of C equals the whole number contained in $\dfrac{A + \sqrt{22}}{B}$.

The values of A and B contained in every line except the first are obtained by means of the formulæ:

$$A_{n+1} = C_n B_n - A_n$$
$$B_{n+1} = \frac{N - A^2_{n+1}}{B_n}.$$

The period closes as soon as $B = 1$.*

Such devices and short cuts exist in large numbers. They have a certain value, when many examples of one type have to be solved for a practical purpose. It is also convenient for the teacher to be acquainted with such methods, but for the pupil they are practically valueless.

The student's memory is unduly taxed, and he is in many cases led to a more mechanical treatment of the subject. The exceptional student who is able to discover such short cuts for himself will derive pleasure

* The symmetries in the above table make it possible to reduce the work to the calculation of one half of the table.

and mental benefit from the *discovery* of the method, but not from the *knowledge* of the same.

Such methods are introduced in schools, usually in order to make the student pass more readily some examination, without consideration of the pedagogic harm done by such an introduction.

Emphasize topics that cultivate the reasoning power. — As pointed out in the beginning of this chapter, there is a certain danger that the study of elementary algebra may become too mechanical. Hence we should attempt to treat every topic so as to encourage reasoning, and to emphasize those topics that require original thinking. The most important of these is undoubtedly the reading problem, but almost any other topic gives some opportunity to make the students think. The omission of some "cases" will also give the student an opportunity to discover by means of his own ingenuity the solutions of problems that are usually solved by a method. This may be done in factoring, and in the solution of simultaneous quadratics.

Emphasize applicable topics. — While large portions of the old-fashioned curriculum in algebra were determined without any guiding principle, most reformers in America and Europe propose in all doubtful cases, to make the applicability of a topic the chief criterion of its importance. If this principle is not carried so far as to sacrifice the serious study of algebra to applications of doubtful value, it certainly deserves approval.

Among the topics which thus deserve special emphasis may be mentioned:

1. Numerical substitution.
2. Equations and problems.
3. Graphs.
4. Proportion.
5. Logarithms.
6. All subjects that lead to the idea of functionality.

The idea of functionality, which is so much emphasized by European writers, has not received the same attention in the United States, partly because here graphic methods have only quite recently been adopted.

General maxims for teaching algebra. — While the preceding discussion relates principally to the curriculum as it finds expression in the textbooks, it has of course a bearing also upon the work of the teacher in the classroom. The principles following from the preceding sections, and a few others, may be summarized as follows:

1. Emphasize all parts of the work requiring original thinking.

2. Emphasize all topics that can be applied, or that lead indirectly to applications; but do not sacrifice the true study of algebra to sham applications.

3. Eliminate as far as possible all merely technical matter that is not necessary for more advanced work. Omit all short cuts and special devices.

4. Omit all theory which is beyond the comprehension of the pupils, or which is logically unsound. If necessary, infer such matters from particular cases, although it will then be necessary to point out to the students that an assumption has been made.

5. By repetition and a certain amount of practical drill, make familiar to the student methods which are of fundamental importance or which are frequently applied.

6. Emphasize the inductive method. Introduce as far as possible every general method by concrete, *i.e.*, numerical, examples. Thus before considering $a^m \times a^n$, find $2^3 \cdot 2^5$; before discovering the relations of the roots of $a_0x^n + a_1x^{n-1} + \cdots a_n = 0$, find the same relation for $x^3 - 4x^2 + 5x - 7 = 0$.

7. Examples that are not of fundamental importance should as far as possible be solved by reasoning and not by memorized methods.

CHAPTER XX

TYPICAL PARTS OF ALGEBRA

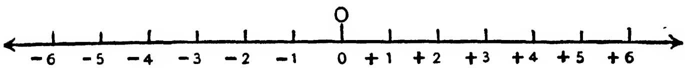
INTRODUCTORY SUBJECTS

The first lessons in algebra. — The method formerly so prevalent, of beginning algebra with a long list of definitions which had to be memorized, is fortunately disappearing. To-day it is quite generally attempted to offer to the students in the first lessons a topic that rouses their interest and that makes them think, as, *e.g.*, applied problems or negative numbers. Simple problems undoubtedly form a good starting topic, although we should bear in mind that this subject is taught here solely for the purpose of justifying the use of letters in place of numbers, and hence we should restrict the work to fairly simple examples. Equally as good a start may be made with negative numbers, a topic that on account of its novelty and simplicity interests students greatly.

Negative numbers. — Negative numbers are usually defined as numbers smaller than zero, and this definition is quite satisfactory from the pedagogical viewpoint, although logically it is objectionable. For the term "smaller than" is usually defined by the statement: a is smaller than b, if $a - b$ is negative. Hence the term "negative" is based upon "smaller than," which in turn rests upon "negative," an obvious circle.

Students have no difficulty in grasping the concept of negative number if the subject is presented concretely on the basis of practical examples. While it is impossible to diminish a group of 5 objects by 7, a temperature of 5° may decrease 7°. If we deal with cardinal numbers, it is impossible to subtract a greater number from a smaller; but when a number is used for measurement, and the quantities considered allow measurement in opposite directions, subtraction is always possible. Quantities of this kind are: gain and loss, latitude, longitude, years A.D. and years B.C., upward and downward motion, opposing forces, temperature, etc.

The most important illustration for further work is the number scale, represented in the annexed diagram.



Practice with a number of examples of these types will not only give to the student a clear understanding of negative number, but will also enable him to perform addition of any two numbers, the subtraction of a positive number from any number, etc., without having studied the laws upon which these examples are based.

The introduction of a special symbol for negative number, as $^{-}4$, in order to discriminate between the sign of quality and the sign of operation, can hardly be recommended. The distinction between the two signs is somewhat artificial, and in some instances it is quite difficult to decide which kind of symbol we have to deal with. Students find this distinction rather difficult and

tedious, and hardly any of the teachers or textbooks that emphasize this matter in the beginning retain it in more advanced work.

Numerical substitutions. — Numerical substitution is one of the most important topics of beginners' algebra. It is the natural link between arithmetic and algebra and insures understanding of algebraic symbols in more advanced work. A student unable to evaluate an algebraic expression for given values of the letters involved cannot possibly possess an understanding of the operations of algebra. On the other hand a student who has learned numerical substitution only has — aside from the purely algebraic gain — acquired an understanding of the meaning and of the use of formulæ, a knowledge which has a distinct practical value.

Numerical substitution should not only be practiced in the beginning, but be applied throughout the course, whenever possible. Students rather easily lose sight of the true meaning of symbols, and make mistakes which they would not make if they realized that these letters represent numbers.*

Hence if a student says $\frac{1}{a}+\frac{5}{a}=\frac{6}{2a}$, let him find $\frac{1}{4}+\frac{5}{4}$, and if he then should answer $\frac{6}{8}$, request the sum of $\frac{1}{4}$ of a dollar and $\frac{5}{4}$ of a dollar. Many stu-

* "Pupils say $\frac{1}{a}+\frac{1}{b}=\frac{1}{a+b}$, who would never say $\frac{1}{3}+\frac{1}{5}=\frac{1}{8}$." (*Lodge.*)

Unfortunately, however, there are high school students who do say $\frac{1}{4}+\frac{5}{4}=\frac{6}{8}$ or $\frac{1}{2}+\frac{1}{2}=\frac{1}{4}$.

dents who say $\frac{x+1}{x}=\frac{1+1}{1}$, would not say $\frac{100+1}{100}=2$, others who say $\sqrt{a^2+b^2}=a+b$, would not say $\sqrt{9+1}=3+1$, etc. Numerical substitution is thus the best means of illustrating emphatically to the students the absurdities of some of their mistakes.

Sometimes, however, the student answers the algebraic question correctly, but makes mistakes when numerical examples of the same type are proposed. Thus a student who knows that $a^3 \times a^4 = a^7$, may say $2^3 \times 2^4 = 4^7$. Such errors show that the pupil has no true insight into the meaning of the algebraic symbols which he uses; in other words, he is manipulating mechanically symbols which he does not understand.

Numerical substitution examples are so important that their practice can hardly be overdone. If the text-book should not contain enough material, the teacher can readily construct examples, whose answers are obvious to him, by using a formula which at this stage of the work is unknown to the pupil. Thus each pupil may substitute different numbers in the expression $a^3 - 3a^2b + 3ab^2 - b^3$, and the teacher can check the results instantly since the answer equals $(a-b)^3$. It is a very simple matter to construct formulæ of this type; a few instances are the following:

$$a^4 - 8a^3b + 24a^2b^2 - 32ab^3 + 16b^4 = (a-2b)^4.$$

$$\frac{a^3+b^3+c^3+3a^2b+3ab^2}{a^2+b^2+c^2+2ab-ac-bc} = a+b+c.$$

$$\frac{a^3}{(a-b)(a-c)}+\frac{b^3}{(b-c)(b-a)}+\frac{c^3}{(c-a)(c-b)} = a+b+c.$$

The student should also practice here the substitution of numerical values in formulæ. Such formulæ may be taken from physics, geometry, or commercial branches. Some very simple formulæ may also be constructed by the pupil, *e.g.*, a formula for the total area of the walls of a room whose dimensions are given.

Checks. — Numerical substitution forms the most widely used and the most convenient means for verifying the answers to algebraic examples. Results of algebraic manipulations cannot be correct unless questions and answers give equal results for all numerical values of the letters involved; and on the other hand, if numerical substitution results in equal values of question and answer, the latter is probably correct.*

As a rule, we select small numbers, but avoid substitutions that lead to forms like $\frac{0}{0}$ or $\frac{4}{0}$. If we make all letters involved equal to unity, we often obtain a very simple test, but since all powers of unity are equal, this method does not check the exponents. Thus to check the multiplication

$$(2a^2 - 3ab - 2b^2)(2a - 3b) = 4a^3 - 12a^2b + 5ab^2 + 6b^3,$$

we let $$a = b = 1,$$

and obtain $$-3 \times -1 = +3.$$

Hence the multiplication is probably correct.

* A single numerical substitution is no absolute test. To make it such several substitutions are necessary, and their number depends upon the nature of the example. Thus if only one letter is involved and the identity $p(x) = \psi(x)$ is of the n^{th} degree, $n + 1$ numerical substitutions constitute an absolute test.

Numerical substitution does not merely indicate whether or not a result is correct, but it may, in case of a wrong answer, also be used to locate the error, as illustrated by the following fallacy:

		Check: Let $x = y = 2$.
Let	$x = y.$	$2 = 2.$
Then	$17x = 17y,$	$34 = 34.$
and	$13x = 13y.$	$26 = 26.$
	$\therefore 17x - 13x = 17y - 13y.$	$8 = 8.$
Or,	$17x - 17y = 13x - 13y.$	$0 = 0.$
	$17(x - y) = 13(x - y).$	$0 = 0.$
	$\therefore 17 = 13.$	$17 = 13.$

Hence the step from the line before the last to the last is erroneous.

A few other methods of verifying algebraic results may be briefly mentioned. Any operation may be checked by the inverse one. Thus, subtraction may be verified by addition, multiplication by division, factoring by multiplication, etc.

The fact that the products, quotients, powers, roots of homogeneous expressions are again homogeneous, and the corresponding fact for symmetric expressions, frequently may be used for checks.

Equations are checked by substituting the roots in both members of the given equation or equations.

ADDITION AND SUBTRACTION

Addition. — The problem of adding positive and negative, or negative and negative, numbers involves a

widening of the definition of addition. Hence the law of addition is really a definition, and not a theorem.

For secondary school purposes it is perfectly legitimate to derive this law from a number of concrete examples, although this is of course no scientific method.*

Subtraction. — Scientifically subtraction is the inverse of addition, or we may define $a - b$ by the equation:

$$(a - b) + b = a.\dagger$$

To the young student, however, subtracting means the taking away of certain things from a group of things, and hence it is advisable to start from this notion and later on lead over to the scientific definition. The taking away of positive numbers from others, that are either positive or negative, is readily explained by gain and loss, northerly and southerly motion, the geometric illustration of the number-scale, etc. To illustrate the subtraction of negative numbers, for instance $(-5)-(-3)$, represent -5 by writing 5 negative units,

$$-1,\ -1,\ -1,\ -1,\ -1,$$

and request the pupil to take away (with the eraser) -3, *i.e.*, three negative units. The result is obviously -2, or $(-5)-(-3) = -2$.

Similarly the total value of the following numbers is 2.

* Compare the Law of No Exception, p. 312.

† All inverse operations can be defined by equations, *e.g.*:

$$\text{Division: } \left(\frac{a}{b}\right)b = a.$$

$$\text{Evolution: } (\sqrt[n]{a})^n = a.$$

$$\begin{array}{r|l} +1 & -1 \\ +1 & -1 \\ +1 & -1 \\ +1 & \\ +1 & \end{array}$$

Taking away -2, the result is 4, or

$$2-(-2)=4, \text{ etc.}$$

Signs of aggregation. — It should be pointed cut to the pupil that the first examples relating to the removal of parentheses are merely additions and subtractions in a new form. Whether we subtract $3a-5b$ from $6a+2b$ by placing the former under the latter, or by writing $(6a+2b)-(3a-5b)$, is simply a matter of arrangement.

It is a waste of time to solve very complex examples of this type. One parenthesis within another is all the student of elementary algebra will have to use in physics, geometry, or trigonometry. Whether we should commence the removal of parentheses from within or from without is a question of no great importance. The latter does not necessitate as many changes of sign as the former, but leads more readily to mistakes. The study of short cuts for the simultaneous removal of several parentheses has no value whatsoever.

MULTIPLICATION

Rule of signs for multiplication. — The multiplication of a negative number by a positive integer follows directly from the definition which considers such a mul-

tiplication a repeated addition. This definition, however, becomes meaningless for a negative multiplier, for, to add a number -7 times is just as meaningless as to read a book -7 times. Hence, the attempts frequently made in the older textbooks to prove that $a \times -b = -ab$, without defining the multiplication by a negative multiplier, were necessarily futile. A proof of this kind is the following:

Since $$(-4) \times 3 = -12,^*$$

and $$ab = ba,$$

$$3 \times (-4) = (-4) \times 3 = -12,$$

i.e., to multiply by a negative number, multiply by its absolute value, and change the sign of the result. But the terms "positive" and "negative" are purely relative.† Hence, we may consider 3 as negative, and 12 as positive, or $$(-3) \times (-4) = 12.$$

This "proof" assumes that $4 \times (-3) = (-3) \times 4$, *i.e.*, it assumes the law of commutation for an operation which has not even been defined. Another "proof" that has occasionally been given is the following:

$$\begin{aligned}(-5) \times (-4) &= (-5) \cdot (3-7) \\ &= (-5) \cdot 3 - (-5)(7) \\ &= -15 - (-35) \\ &= 20.\end{aligned}$$

* The symbol $\times$ stands here for "multiplied by." If read "times" the sequence of factors should be altered.

† This becomes obvious if we consider the equation 3° N. $\times (-4) = 12^\circ$ S. If we consider N. as positive, we have $3 \times -4 = -12$; if we consider N. as negative, the equation means $(-3)(-4) = 12$

Again it is assumed that products containing negative multipliers are subject to the same law (distribution) as those containing positive multipliers.

The more rigorous books introduce the following definition of multiplication: Multiplication is the operation of finding a number that has the same relation to one factor (multiplicand) as the other factor (multiplier) has to unity. But, aside from the vagueness of the term "relation," this definition is too difficult for beginners.

Pedagogically it seems best to attack at first practical problems, and then to introduce a definition that agrees with the results of the concrete examples. Thus we may consider a ship sailing north at the rate of $2°$ per day, and crossing the equator to-day; 5 days hence the ship will be at $10°$ N., or $2 \times 5 = 10$. 5 days ago the ship was at $10°$ S., or $2 \times (-5) = -10$. A ship sailing south under the same conditions leads to $(-2) \times 5 = 10$ and $(-2) \times (-5) = 10$.

Or we may consider the (opposing) forces produced by adding and taking away of a number of equal weights at the two sides of a balance. We may consider the changes in the lifting power of a balloon produced by increasing or decreasing the quantity of gas or the ballast. We may consider the changes in the income of a town by the arrival and departure of taxpayers and alms receivers, the former paying, the latter receiving, \$100 annually.

It appears from such examples that it is *convenient* to define multiplication by a negative multiplier as a

repeated subtraction, or $(-4) \times (-3) = -(-4) - (-4) - (-4)$.

Having adopted this definition, the pedagogy of the subject of multiplication offers no more difficulty.

Law of No Exception. — In the preceding paragraph it was shown that a multiplication by a negative multiplier requires a widening of the original definition. Such modifications of definitions occur a number of times in algebra. While for school purposes it is legitimate to derive such matters from applications, we should bear in mind that scientifically, algebra is what it is, not in virtue of its applications, but in virtue of certain fundamental laws. Hence applications cannot demonstrate anything in algebra.

The scientific principle that guides us in such generalizations, and that has been called the Law of No Exception, or the Principle of the Permanence of Equivalent Forms may be stated as follows:

"In the construction of arithmetic, every combination of two previously defined numbers by a sign for a previously defined operation ($+$, $-$, $\times$, etc.) shall be invested with a meaning, even where the original definition of the operation excludes such a combination, and the meaning imparted is to be such that the old laws of reckoning shall still hold good." (Schubert.)

Thus the definition of multiplication must be widened in such a way that we may operate with negative numbers in precisely the same way as with positive numbers, or in other words: A calculation involving *a*, *b*, *c*,

etc., shall be just the same whether a, b, c, etc., represent positive or negative numbers. This is accomplished by determining the definition so that it conforms to the fundamental laws of multiplication, viz., the commutative, the associative, and the distributive law.

Similarly the widening of any definition of any operation must conform to the fundamental laws of this operation. This may be further illustrated by considering the definition of power.

The original definition of a^n, when n is a positive integer, becomes absolutely meaningless when n is negative or fractional. Hence, it is impossible to prove that $a^{\frac{1}{3}} = \sqrt[3]{a}$. We have to widen the definition, and there is nothing in the original definition that compels us to take a definite course. It would not be *illogical* to define $a^{\frac{1}{n}}$ by $\frac{a^n}{n}$, but it would be very *inconvenient*, for this would lead to entirely different laws for fractional than for integral exponents. Hence, we could never perform a calculation involving exponents unless we knew whether n were integral or fractional. To avoid this we have to determine which definition of fractional exponents leads to fundamental laws that are identical with those of positive integral exponents. (These laws are: $a^m \cdot a^n = a^{m+n}$, $a^m \div a^n = a^{m-n}$, $(a^m)^n = a^{mn}$, $(ab)^m = a^m b^m$.)

Thus we are led to the definitions: $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ and $a^{-n} = \frac{1}{a^n}$.

FACTORING

When to study factoring. — Since each mode of factoring is based upon a method of multiplication, students will most readily discover a method of factoring at the time when the corresponding multiplication is studied. Thus after studying multiplication examples of the type $(x+a)(x+b)$, pupils will easily discover the factors of $a^2+3a-10$. It is therefore advisable to attach to most sets of multiplication examples some factoring questions. This, however, does not imply that the methods of factoring should afterwards not be collected, and presented connectedly. A second and connected presentation of all cases in factoring will impress these important facts firmly upon the student's mind, it will lead to a comparison of the various methods, to proper notions about the selection of a method for a certain purpose, and it will make reviews easier. Hence, it seems to the writer that the plan of connecting factoring and multiplication should not be carried out to the exclusion of a formal and systematic presentation of the subject later on.*

* The idea that every algebraic topic should be taught in connection with the subjects with which it is logically connected cannot be carried out in practice, since the number of such connections is too large. Thus the multiplication $a(b+c)=ab+ac$ is connected with the factoring example $ab+ac=a(b+c)$, but also with the division $\frac{ab+ac}{a}=b+c$, and this again leads to addition of fractions: $\frac{ab}{a}+\frac{ac}{a}=\frac{ab+ac}{a}$. But obviously all these topics cannot be taught with $a(b+c)$.

Which cases of factoring should be studied? — For most students factoring forms one of the most difficult chapters of elementary algebra. The study of factoring sometimes discourages pupils so much that they lose all interest in the subject, and become indifferent and inefficient students of algebra in general. Hence we should try to represent factoring as simply as possible, and to exclude all cases that are not necessary for future work.

Until quite recently the policy of most textbook writers, and of most schools, however, has been to work in exactly the opposite directions. Instead of presenting a few cases and practicing them thoroughly, every case that possibly could be dragged in was taught. Through extended mechanical drill and memorizing students were enabled to work quite difficult examples at the time when the subject was studied. But a short time afterward all was forgotten, and since the students were unable to reconstruct any method, their knowledge of factoring was very small in spite of the long-continued drill.

The cases which are absolutely necessary are of the following types:

1. $ax + ay + az$.
2. $ax^2 + bx + c$, and the special cases

   $a^2 \pm 2ab + b^2$, and

   $x^2 + px + q$.
3. $a^2 - b^2$.

At a later stage of the work these may be studied:

4. $a^3 \pm b^3$.
5. Grouping terms.
6. The Remainder Theorems.

Superfluous cases. — The superfluity of other methods may be illustrated by a discussion of the five cases which relate to $a^n \pm b^n$, and which are sometimes stated as follows:

1. $a^n - b^n$ is divisible by $a - b$, if n is even.
2. $a^n - b^n$ is divisible by $a + b$, if n is even.
3. $a^n + b^n$ is not divisible by $a + b$, if n is even.
4. $a^n - b^n$ is divisible by $a - b$, if n is odd.
5. $a^n + b^n$ is divisible by $a + b$, if n is odd.

Now, first of all, we never use the factors $a + b$ or $a - b$ directly in examples of the type $a^n \pm b^n$, unless *n is prime.*

We do *not* factor

$$a^{15} + b^{15} = (a + b)(a^{14} - \cdots),$$
$$a^6 - b^6 = (a - b)(a^5 + \cdots),$$

But we let:

$$a^{15} + b^{15} = (a^5)^3 + (b^5)^3 = (a^5 + b^5)(a^{10} - \cdots),$$
$$a^6 - b^6 = (a^3)^2 - (b^3)^2 = (a^3 - b^3)(a^3 + b^3).$$

In other words we try to represent the two given powers first as *two squares*, then as *two cubes*, then as *two fifth powers*, etc.; *i.e.*, we consider only *powers whose exponents are prime.*

But the only even prime number is 2, hence all we

need to know about even powers is contained in the two well-known facts:

$$a^2 - b^2 = (a - b)(a + b). \tag{1}$$

$$a^2 + b^2 \text{ is prime.} \tag{2}$$

Hence the first three rules are absolutely superfluous. But they are worse than that: they are misleading. To select $a - b$ as a factor of $a^{12} - b^{12}$ is the worst thing the student could do. *If n is even, $a^n - b^n$ should always be considered the difference of two squares.* Thus

$$a^{12} - b^{12} = (a^6 - b^6)(a^6 + b^6)$$

and not $\qquad (a - b)(a^{11} + \cdots).$

Similar considerations for odd powers show us that these formulæ should not be used except for prime values of n, *i.e.*:

$$a^3 \pm b^3,$$
$$a^5 \pm b^5,$$
$$a^7 \pm b^7,$$
$$a^{11} \pm b^{11}, \text{ etc.}$$

Of these $a^3 \pm b^3$ is usually studied independently, and before $a^n \pm b^n$ is considered. But as hardly anybody advocates the factoring of $a^7 \pm b^7$ or higher powers, the student has to study five complex — and, on account of their similarity, very confusing — rules, for the noble purpose of factoring $a^5 \pm b^5$.*

* Of course higher powers are sometimes used in advanced mathematics, as, *e.g.*, in the study of binomial equations, and the connected topics of constructing regular polygons. To construct a regular polygon of seventeen sides we have to factor $x^{17} - 1$. But how many pupils in beginners' classes will ever study the construction of the regular polygon of seventeen sides!

The instructor who wishes to teach all examples relating to the binomial $a^n \pm b^n$ has to teach no other formula than $a^5 \pm b^5$, which should be done concretely and not by having memorized the above five rules.

Factoring $ax^2 + bx + c$.—One rather indispensable case, about whose presentation a wide diversity of opinion exists, is the factoring of the quadratic trinomial $ax^2 + bx + c$. At a more advanced stage of the work it is shown that $ax^2 + bx + c = a(x - r_1)(x - r_2)$, where r_1 and r_2 are the roots of the equation $ax^2 + bx + c = 0$. For beginners this method may be represented by a concrete example as follows:

$$\begin{aligned} x^2 - 3x - 4 &= x^2 - 3x + (\tfrac{3}{2})^2 - (4 + \tfrac{9}{4}) \\ &= (x - \tfrac{3}{2})^2 - (\tfrac{5}{2})^2 \\ &= (x - \tfrac{3}{2} + \tfrac{5}{2})(x - \tfrac{3}{2} - \tfrac{5}{2}) \\ &= (x + 1)(x - 4). \end{aligned}$$

But even in this form most examples are too difficult for the beginner.

Another method makes every quadratic trinomial a special case of $x^2 + px + q$, as illustrated by the following example:

$$\begin{aligned} 3x^2 - 10x + 3 &= \frac{9x^2 - 30x + 9}{3} \\ &= \frac{(3x)^2 - 10(3x) + 9}{3} \\ &= \frac{(3x + 9)(3x - 1)}{3}, \text{ etc.} \end{aligned}$$

This method, however, requires fractions which are not known to the students at this stage of the work. Since

it also leads to very large numbers, it is not surprising that it does not work well in most classes.

Another method which has been praised a great deal by some pedagogues splits the coefficient of x into two numbers, thus, that their product equals ac. *E.g.*, to

factor $$6x^2 - 95x + 75,$$

find two numbers whose sum is -95 and whose product is 450. These numbers are -5 and -90.

$$\begin{aligned} 6x^2 - 95x + 75 &= 6x^2 - 90x - 5x + 75 \\ &= 6x(x - 15) - 5(x - 15) \\ &= (6x - 5)(x - 15). \end{aligned}$$

There are two objections to this method. The numbers become sometimes exceedingly large, and the student is unable to understand the reasons for the procedure.

The most natural way, and also the way that produces the best results in most classes,* considers the operations the reverse of the corresponding multiplication. This method, usually called cross-product method, has been frequently attacked by writers who called it a method of guessing, but who seem to forget that the other methods are also based upon guessing.

The cross-product method becomes comparatively simple if we free the given expression from monomial

* Since the above was written, Mr. Fiske Allen has reported on some experiments which tested the three different methods in different classes (*Mathematics Teacher*, September, 1911). According to these tests the cross-product method was by far the most effective.

factors, and bear in mind that each of the resulting binomial factors cannot then contain a monomial factor. Thus to factor

$$72x^2 - 145x + 72.$$

We may factor $72x$ in many different ways, but obviously these two factors must not have a common factor, since otherwise we would obtain a monomial factor in one of the binomials. Hence we have to try only $72x \cdot x$ and $9x \cdot 8x$. Excluding all numbers that produce monomial factors, the last term can be factored only $1 \cdot 72$ and $8 \cdot 9$.

Hence we have only two possibilities:

$$\begin{array}{rr} 72x - 1 \qquad & 9x + 8 \\ x - 72 \qquad & 8x - 9 \end{array}$$

The first combination obviously gives too large a middle term, while the second produces $-145x$. Therefore

$$72x^2 - 145x + 72 = (9x - 8)(8x - 9).$$

A few difficult cases. — While it is unpedagogic to crowd the pupil with too many methods, the teacher should be acquainted with all the more important methods in order to be equal to any emergency. A method that is very interesting and that solves many examples which otherwise would be exceedingly difficult, is based upon symmetry and cyclo-symmetry.

A function is symmetric with respect to x and y if an exchange of these letters does not change the function, *e.g.*, $3x^2 - 5xy + 3y^2$. A function is symmetric with

respect to x, y, and z, if it is symmetric with respect to any two of these letters, *e.g.*,

$$x^2 + y^2 + z^2,\ a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2.$$

A function is cyclo-symmetric with respect to x, y, and z, if by replacing x by y, y by z, and z by x, we obtain an identical equal function, *e.g.*,

$$(x - y)^2 + (y - z)^2 + (z - x)^2$$

$$a^2b + b^2c + c^2a.$$

Every symmetric function is cyclo-symmetric, but not *vice versa*.

The symmetric functions of the first three degrees are

(1) $a(x + y + z)$.

(2) $a(x^2 + y^2 + z^2) + b(xy + yz + zx)$.

(3) $a(x^3 + y^3 + z^3) + b(x^2y + xy^2 + y^2z + yz^2 + x^2z + xz^2) + cxyz$.

The factoring of symmetric and cyclo-symmetric expressions is illustrated by the following examples:

Ex. 1. Factor $x^2(y-z) + y^2(z-x) + z^2(x-y)$.

Since the substitution $x = y$ reduces the function to zero, the function is divisible by $x - y$. But the function is cyclo-symmetric, hence it is also divisible by $y - z$ and $z - x$. Since the given function is homogeneous and of the third degree, the factors $(x - y)$, $(y - z)$, and $(z - x)$ are all its literal factors, but there may be an unknown numerical factor. If k be this numerical factor, we have

$$x^2(y-z) + y^2(z-x) + z^2(x-y) = k(x-y)(y-z)(z-y).$$

Substituting numerical values for x, y, and z, e.g. $x = 2, y = 1, z = 0$; $4 - 2 + 0 = k(-2)$, or $k = -1$.*

Hence

$$x^2(y-z)+y^2(z-x)+z^2(x-y)=-(x-y)(y-z)(z-x).$$

Ex. 2. Factor $x(y-z)^3+y(z-x)^3+z(x-y)^3$.

If $x = y$, the function vanishes, hence it is exactly divisible by $x-y$. Since the function is cyclo-symmetric, $y-z$ and $z-x$ are also factors of the function.

Since the given function is homogeneous and of the fourth degree, there must be another factor of the first degree which is cyclic and homogeneous. Such factors are of the form $k(x+y+z)$. Hence

$$x(y-z)^3+y(z-x)^3 \times z(x-y)^3$$
$$=k(x-y)(y-z)(z-x)(x+y+z).$$

Making $x = 2, y = 1, z = 0$,

$$2-8+0=k(-6), \text{ or } k=1.$$

Hence

$$x(y-z)^3+y(z-x)^3+z(x-y)^3$$
$$=(x-y)(y-z)(z-x)(x+y+z).$$

Ex. 3. Factor $(x-y)^5+(y-z)^5+(z-x)^5$.

In like manner as in the preceding examples we obtain the factor $(x-y)(y-z)(z-x)$. The remaining

* The value of k may also be formed by comparing the coefficients of similar terms in both members. Thus $x^2y = -kx^2y$, i.e., $k = -1$.

factor must be of the second degree and cyclo-symmetric, *i.e.*, it must be of the form

$$k(x^2+y^2+z^2)+l(xy+yz+zx),$$

where k and l are unknown numbers. Hence

$$(x-y)^5+(y-z)^5+(z-x)^5$$
$$=(x-y)(y-z)(z-x)[k(x^2+y^2+z^2)+l(xy+yz+zx)].$$

Making $x=2$, $y=1$, $z=0$, we obtain:

$$5k+2l=15. \tag{1}$$

Making $x=1$, $y=0$, $z=-1$, we obtain:

$$2k-l=15. \tag{2}$$

Solving (1) and (2),

$$k=5,\quad l=-5.$$ Hence

$$(x-y)^5+(y-z)^5+(z-x)^5$$
$$=5(x-y)(y-z)(z-x)(x^2+y^2+z^2-xy-yz-zx).$$

The following list contains a few examples of this type with their answers; others may be found in any of the larger treatises on algebra.

1. $(x-y)^3+(y-z)^3+(z-x)^3=3(x-y)(y-z)(z-x).$
2. $(x+y+z)^3-x^3-y^3-z^3=3(x+y)(y+z)(z+x).$
3. $xy(x-y)+yz(y-z)+zx(z-x)$
$$=-(x-y)(y-z)(z-x).$$
4. $x^3(y-z)+y^3(z-x)+z^3(x-y)$
$$=-(x-y)(y-z)(z-x)(x+y+z).$$
5. $x^4(y^2-z^2)+y^4(z^2-x^2)+z^4(x^2-y^2)$
$$=(x+y)(x-y)(y+z)(y-z)(z+x)(z-x).$$

6. $x^4(y-z)+y^4(z-x)+z^4(x-y)$
$=-(x-y)(y-z)(z-x)(x^2+y^2+z^2-xy-yz-zx)$.

7. $(x+y+z)^5-x^5-y^5-z^5$
$=5(x+y)(y+z)(z+x)(x^2+y^2+z^2+xy+yz+zx)$.

8. $(x+y)(y+z)(z+x)+xyz=(x+y+z)(xy+yz+zx)$.

EQUATIONS AND PROBLEMS

Identities and equations of condition. — There is obviously a great difference between the statements $a+a=2a$, and $a+7=2a$. The former, called an *identity*, is true for all values of a; the latter, called an *equation*, is true only for a certain value of a, viz., 7. An identity, sometimes also called an identical equation, states a demonstrable mathematical fact; it is a theorem. An equation (or equation of condition) requires the finding of the root or roots, hence it is a problem. Equations must be solved; identities must be proved.

It is not always possible to decide by inspection whether an equality is an identity or an equation of condition, but this fact appears if we apply the usual method for solving equations. If all terms cancel, *i.e.*, if the equality reduces to the form $0=0$, it is an identity. Thus, to decide whether the following statement is an identity or an equation we proceed as follows:

$$\frac{1}{x}-\frac{x+3}{6x}=\frac{x+3}{6x}-\frac{1}{3}.$$

Clearing of fractions, $\quad 6-x-3=x+3-2x.$

Transposing, $\quad -x-x+2x=-6+3+3,$

or $\quad 0=0.$

The result indicates that the given statement is an identity, and the above deduction is a proof of the identity.

If not every term cancels, the equality is an equation. It may happen, however, that an equation is not satisfied by any finite value of the unknown quantity, *e.g.*,

$$x + 3 = x + 4.$$

Since $\infty + 3 = \infty + 4$, we may say the root $x = \infty$.

If in an equation all terms containing the unknown quantity cancel, while the remaining terms do not cancel, the root is infinity. In case of an applied problem, however, the root infinity indicates that the problem has no solution.

Similar remarks may be made about two equations involving two letters, *e.g.*, x and y. If we eliminate one of the two quantities, the resulting equality (called the eliminant) may be an identity. In such a case each of the given equations is a consequence of the other. The equations are *dependent*, and any value whatsoever may be assigned to one of the unknown quantities. Thus, if we eliminate y by comparison from the following system:

$$\begin{cases} y = \dfrac{2x+3}{x^3 - 7x - 6} + \dfrac{x+1}{x^2 - x - 6}, & (1) \\ y(x^2 - 2x - 3) = x + 2, & (2) \end{cases}$$

we obtain:

$$\frac{2x+3}{x^3 - 7x - 6} + \frac{x+1}{x^2 - x - 6} = \frac{x+2}{x^2 - 2x - 3}. \qquad (3)$$

The solution of (3) leads to the result $0 = 0$. Hence equations (1) and (2) are dependent.

If the eliminant leads to an equality of the form $0 \cdot x = a$, no finite values of the unknown quantities satisfy the equations. Such equations are called *inconsistent*. A system obviously inconsistent is the following:

$$\begin{cases} 3x + 6y = 7, \\ 3x + 6y = 8.^* \end{cases}$$

The preceding topics have little value for the pupil in a secondary school, but the teacher should be acquainted with them, since occasionally questions of this type will arise in the classroom.

Equivalent equations. — Suppose in the following equation we should not recognize the L. C. D., and multiply both members by $2(x-1)(x+1)$, we should obtain the following solution:

$$\frac{4}{x-1} = \frac{3x+2}{2x-2} \qquad (1)$$

$$8(x+1) = (3x+2)(x+1),$$

or

$$8x + 8 = 3x^2 + 5x + 2. \qquad (2)$$

Hence $$3x^2 - 3x - 6 = 0. \qquad (3)$$

$$x^2 - x - 2 = 0. \qquad (4)$$

Therefore, $$x = -1, x = 2. \qquad (5)$$

But only one of these roots, viz., $x = 2$, satisfies (1), and the question arises what error produced the answer $x = -1$.

Quite often equations are treated as if the given equation (1) was true and we had to *prove* that each

* Graphic methods explain most lucidly the nature of inconsistent, consistent, independent, and dependent equations.

successive equation (2, 3, 4, etc.) was true. As each of these statements follows from the preceding one by a certain axiom, no error could thus be found in the above example. However, we do not have to prove a theorem, but to find a root that satisfies (1); *i.e.*, we must show that (1) is true, if (5) is true, or we have to examine the steps from (5) to (4), from (4) to (3), etc. The step from (2) to (1) is justified by the axiom: If equals be divided by equals, the quotients are equal. This axiom, however, is not true for zero divisors. Hence this step is not justified if

$$2(x+1)(x-1)=0, \text{ i.e., } x=-1, \text{ or } x=+1.$$

Hence the value $x=-1$, which satisfies equation (2), does not need to (and this example does not) satisfy equation (1).

Multiplying both members of an equation by an expression involving the unknown quantity usually introduces a new root (called an *extraneous* root).

Thus $\qquad x-3=4$ has one root, $x=7$.

$(x-3)(x-5)=4(x-5)$ has two roots,

$x=7$ and $x=5$.

Equations which have the same roots are called *equivalent equations*. In the solution of an equation it is necessary to prove that every equation is equivalent to the preceding one. In elementary algebra, however, the solutions of equations lead only very rarely to equations that are not equivalent to the given one. Two cases of this kind are the following:

Multiplying the members of a fractional equation by a multiple of the denominators, which is not the lowest, introduces extraneous roots.

Squaring both members of equations usually introduces extraneous roots; *e.g.*,

If $$x - 4 = 2, \tag{1}$$

then $$x^2 - 8x + 16 = 4. \tag{2}$$

Equation (1) has only one root, viz., $x = 6$, while equation (2) has two roots, $x = 2$ and $x = 6$.

Consequently the solutions of radical equations often lead to extraneous roots,* and all roots of radical equations require checking.

Quadratic equations. — The most important method for solving quadratics is the one based upon the formula. The method based on completing the square is necessary for obtaining this formula and for reconstructing it, if the student should forget. Completing the square, however, is not well adapted to the solution of more complex numerical or of literal equations, and it involves in any example a good deal of unnecessary labor. Hence while this method deserves some practice, it should not be extended too far, and certainly not to the study of several methods for completing the square.

Since the formula for the roots of the equation $x^2 + px + q = 0$ leads to complex fractions in case the coefficient of x^2 is not unity, it is better to study the

* This happens, however, only if we restrict the values of the radicals to their prefixed signs.

formula for the roots of the equation $ax^2 + bx + c = 0$, viz.:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This formula should be thoroughly memorized and applied to many numerical and literal examples.

The third method which deserves serious attention is the solution by factoring. Many writers recommend this method, on account of its simplicity, as the first method to be studied. If factoring could be used for the solution of every equation, and the formula could thus be entirely excluded, this would undoubtedly be the best plan. But students in order to solve all examples must study the formula, and if this is done after equations have been solved by factoring, such work is usually considered very tiresome and unnecessary. On the other hand, students who first study the completing of the square usually take great interest and pleasure in the study of the factoring method. It is a revelation to them that examples which formerly required so much work can be done so simply and so quickly. Moreover, we must not forget that in practical examples the coefficients depend upon measurement, and hence practically never lead to equations that can be factored.

Besides simplicity, the method based on factoring has two advantages, viz., it can be applied to equations of higher degree, and it produces *all* roots of an equation more readily than the other methods.

In solving $x^6 - 9x^3 + 8 = 0$ by the formula, the student obtains $x^3 = 8$ and $x^3 = 1$. Hence he is likely

to conclude: $x = 2$, $x = 1$. But the factoring method readily leads to six roots because

$$x^6 - 9^3 + 8 = (x - 2)(x^2 + 2x + 4)(x - 1)(x^2 + x + 1).$$

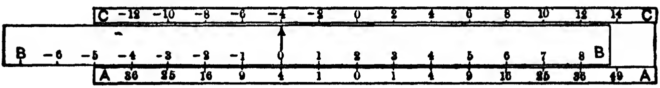
The factoring method also shows us that dividing both members of an equation by an expression involving x removes one or more roots. Thus, when dividing both members of the equation $x^2 - a^2 = (x - a)(a + b - x)$ by $x - a$, we should note that one root is obtained by making $x - a = 0$, while the other follows from

$$x + a = a + b - x.^*$$

Applied problems. — Applied problems, or, as they are often called, reading problems, form possibly the most important topic of elementary algebra. Unfortunately,

* Among the mechanical devices for solving quadratics, there may be mentioned a slide rule which the author has constructed.

It consists of a rule bearing the scale B, which slides in another rule bearing scales A and C. The scale B is the natural scale of positive and negative numbers, the numbers on the scale C are the doubles of the corresponding ones on B, and the numbers on A are the squares of those



on B. Thus on A 2 units from 0, we find 4; 3 units from 0, we find 9, etc.

To solve $x^2 + px + q = 0$ (in the diagram $x^2 - 4x - 32$) put the arrow 0 below the number p (*i.e.*, -4) in C, and read off the corresponding number in A (*i.e.*, 4). From this number subtract the absolute term (-32), and locate the difference (36) in scale A. Opposite these numbers (36) in scale B are found the roots:

$$x = -4, \text{ and } x = 8.$$

This slide rule can be used to find large and small roots, imaginary roots, maxima or minima of the function, etc.

many students find problem work exceedingly difficult and consequently loathe this subject. The ability to solve problems requires two things, viz., the ability to think, and a knowledge of the technique of such work. Many students who can reason logically fail because they have not grasped the technique of the subject. Hence we should try to introduce students systematically and slowly into the methods of attacking such work.

Every problem presupposes the ability to translate an English sentence into algebraic shorthand, and a systematic study of such translations should form the starting point for the study of problems. At first the student should write in algebraic symbols expressions like the following:

The sum of the squares of a and b.

The product of the cubes of a and b.

The cube of the difference of m and n.

Then should follow translations that have a bearing upon problems, as:

By how much does a exceed 10?

Write three consecutive numbers whose smallest is x.

A is 20 years old. How old will he be in x years hence?

Find x % of 700.*

As far as possible, all quantities that later on occur in problems and their relations should be considered here. The next step is writing equations without at-

* If such questions should be too difficult, propose the corresponding arithmetic questions; *i.e.*, By how much does 12 exceed 10? etc.

tempting to solve them. The first equations should be so simple that they can be translated, word after word, *e.g.*,

The double of a equals 10.

$$2 \quad \times a \quad = \quad 10.$$

Which number is 5 % of 450?

$$x \quad = \tfrac{5}{100} \times 450.$$

After a fairly good amount of practice, reading problems may be attacked, — at first only such as refer to one unknown number and as can be translated directly; then those which involve two unknown quantities, where one sentence is used to express one unknown quantity in terms of the other, while the other sentence produces the equation.

In a similar way we should in all following chapters on problems attempt a complete classification of the examples and avoid the accumulation of several difficulties. The details of such a method can, however, be fully explained only in a textbook.*

It is true that by no means every problem can be made a special case of a certain type. But by studying simple problems arranged according to a certain system, the student acquires a familiarity with the technique of the subject which in many cases will help in the solution of problems that require real original thinking.

GRAPHS

Reasons for teaching graphs. — The reasons that have led to the introduction of graphs into our secondary school courses may be summarized as follows:

* See the author's Algebra.

1. The study of graphs is very concrete and hence counteracts somewhat the tendency of school algebra to become a mechanical application of known rules.

2. Graphic representations are at present so widely used in daily papers, magazines, and books, that a certain familiarity with these devices is a part of general culture.

3. This mode of representing variables is used a great deal in other sciences. To study successfully physics, mechanics, chemistry, meteorology, economics, etc., the student has to be acquainted with graphs.

4. By graphs many mathematical facts become visible to the eye, which otherwise would remain obscure, *e.g.*, the number of roots of various equations, the nature of inconsistent equations, of independent equations, etc.

5. The study of graphs enables the student to solve many examples which he could otherwise not solve at all; as, the solution of higher equations, the solution of transcendental equations, etc.

6. The student acquires a clear notion of one of the most important notions of advanced mathematics, viz., functionality.

7. Graphs interest students and are easily understood.

Introductory examples. — The teaching of the fundamental ideas of graphs is so simple, and the material for such work so abundant, that only a few points need be mentioned here:

1. Classify the work as follows:

a. Graphic representation of a given numerical

table; as, graphs of temperature, population, etc. (See U. S. Statistical Abstract, or the World Almanac.)

b. Graphs of numerical tables which the student has to calculate; *e.g.*, the cost of iron from 0 lb. to 6 lb.

c. Graphs of physical and geometric formulæ; as, $C = \frac{44}{7} \cdot R$.

d. Algebraic graphs.

e. Graphic solution of problems.

2. It is not sufficient to construct graphs; they should also be interpreted. Thus in a temperature graph, the student should find the temperature at a given time, the time corresponding to a given temperature, maxima and minima, the time of most rapid increase of temperature, etc.

3. Students should learn that the graph is a straight line through the origin if the two quantities involved are proportional; and should derive therefrom a quick method for constructing such graphs.

4. To obtain fairly good results it is necessary to use cross-section paper. An ordinary ruled sheet can also be used to advantage. For blackboard work a system of squares may be scratched into the board by a triangular file (make side equal to about 2 inches).

5. Do not spend too much time in constructing statistical graphs.

Graphs of functions. — It is not sufficient that students construct the graphs of functions, and thereby solve

equations; they should use these diagrams to solve many other problems relating to functions and equations. Thus students should find maxima and minima; they should recognize that the same drawing may be used to solve $f(x) = 0$, and $f(x) = 5$; they should see why $x^3 - 6x^2 + 11x - 6 = 0$ has three roots, while $x^3 - 6x^2 + 11x - 6 = 40$ has only one root, etc.

In the discussion of simultaneous equations, graphs may be used to show why simultaneous linear equations have only one set of roots, while those of higher degree have several sets. Graphs make clear to the student why inconsistent linear equations have no finite root, and why dependent equations have an infinite number of roots.

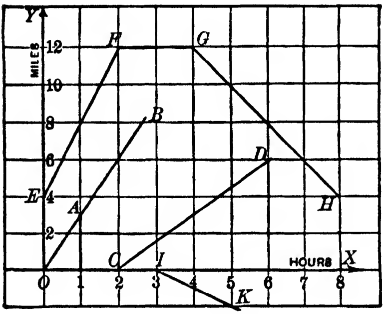
Graphic solution of problems. — Problems are usually solved in algebra by expressing the conditions of the problems in the form of equations. By using the graphic method, however, many problems can be solved directly, without obtaining equations.

The fact mentioned above, that the graph of two proportional variables is a straight line, is often useful. Thus, if x and y are the coördinates of a point, the following variables are represented by straight lines: $x =$ time, $y =$ distance covered by body moving uniformly; $x =$ time, $y =$ work done by a person; $x =$ volume, $y =$ weight of a body; $x =$ time, $y =$ quantity of water flowing through a pipe at a uniform rate, etc.

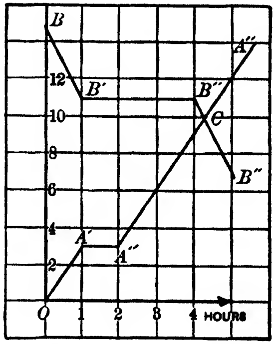
To represent graphically the motion of a person traveling 3 miles per hour, it is only necessary to locate one point, *e.g.*, (1, 3) or A, and to connect this point to

the origin. The increase of the ordinate per hour equals the rate of travel, *i.e.*, 3 miles per hour.

Similarly, *CD* represents the motion of another person who started 2 hours later and traveled $1\frac{1}{2}$ miles per hour. *EFGH* represents graphically that a third person had a start of 4 miles, traveled 2 hours at the rate of 4 miles per hour, then rested 2 hours, and finally returned to the starting point at the rate of 2 miles per hour. *IK* represents graphically the motion of a fourth person who started 3 hours after the first and traveled in the opposite direction at the rate of 1 mile per hour.



Ex. 1. A and B start walking from two towns 15 miles apart, and walk toward each other. A walks at the rate of 3 miles per hour, but rests 1 hour on the way; B travels at the rate of 4 miles per hour and rests 3 hours. In how many hours do they meet?

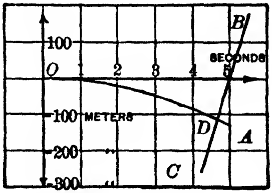


Construct the graphs $OA'A''A'''$ and $BB'B''B'''$. The abscissa of C, the point of intersection, is the required time.

Hence A and B meet in $4\frac{1}{4}^{+}$ hours.

Ex. 2. A stone is dropped into a well, and the sound of its impact upon the water is heard at the top of the well 5 seconds later. If the velocity of sound is assumed as 360 meters per second, and $g = 10$ meters, how deep is the well?

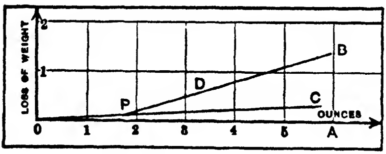
(A body falls in t seconds $\frac{g}{2}t^2$ meters.)



Construct the graph *ODA* of the falling body, making the distances negative, to indicate the downward motion. Since the motion of the sound is an upward motion, its graph *CB* is obtained by joining (4, − 360) and (5, 0). The ordinate of the point of intersection *D* is the required number.

Hence depth of well = 110 meters.

Ex. 3. Six ounces of gold quartz lose $1\frac{1}{2}$ ounces when weighed in water. If quartz loses $\frac{1}{3}$ of its weight, and gold $\frac{1}{19}$ of its weight when placed in water, how many ounces of gold are in the six ounces?

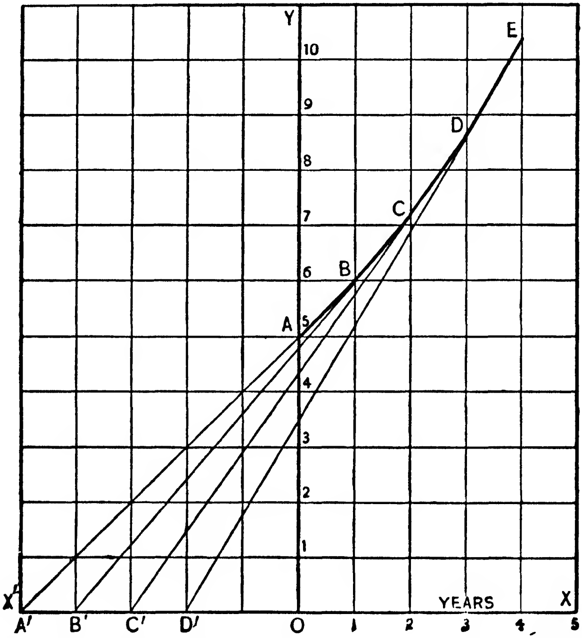


Let *OA* represent the weight of the body, and *AB* the loss of weight in water. Through *O* draw *OC*, ascending at the rate of $\frac{1}{19}$ (*i.e.*, joining *O* and (19, 1), or *O* and (5, $\frac{5}{19}$)). Through *B* draw *DB*, ascending at the rate of $\frac{1}{3}$. The abscissa of *P*, the point of intersection of *OC* and *BD*, represents the required amount of gold ($1\frac{3}{4}$ ounces).

Ex. 4. Find the amount of \$5 for 4 years at 20% compound interest.*

*The high rate of interest is taken in order to produce a small diagram.

Represent the year by the unit of abscissas and the dollar by the unit of the ordinates, and let OA represent \$ 5.00. On OX' lay off $OA' = \frac{100}{\text{rate}} = \frac{100}{20} = 5$. Draw $A'A$ and produce it to B, then the ordinate of B represents the amount after one year. Draw $B'B$ and



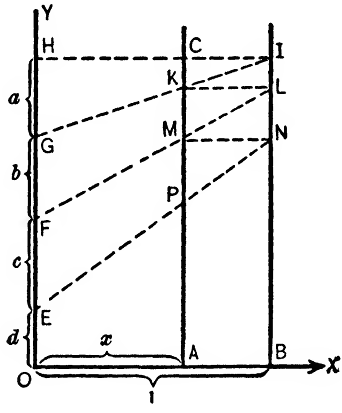
produce to C, then C represents the amount after two years, etc. The ordinate of E, *i.e.*, \$10.30, represents the required amount.

NOTE. — It is of course not necessary to draw the lines AA', BB', etc. Place the ruler so that the prolongation of BA will pass through A', etc. The broken line $ABCDE$ represents the amount better than the formula $a(1 + r)^n$, since the latter is true only for integral values of n.

Purely graphic methods. — The customary method of constructing the graphs of a function is not purely

graphic, since it requires considerable arithmetical work, viz., the calculation of the values of the ordinates. These values, however, may be found by constructions, and thus the work made purely graphic. The construction of values of a rational integral function may be illustrated by constructing the value of the cubic function $ax^3 + bx^2 + cx + d$ for a given value of x.

On OY lay off $OE = d$, $EF = c$, $FG = b$, $GH = a$, and make $OA = x$, $OB = 1$. Draw AC and BI parallel to OY.



Draw $HI \parallel OX$, and let GI meet AC in K.

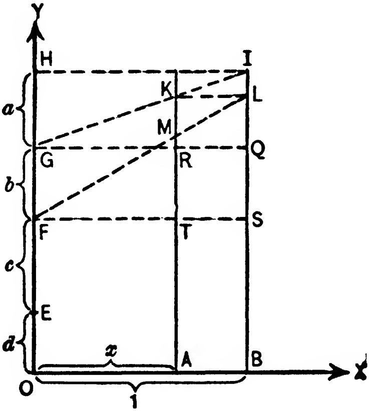
Draw $KL \parallel OX$, and let LF meet AC in M.

Draw $MN \parallel OX$, and let NE meet AC in P.

Then $AP = ax^3 + bx^2 + cx + d$.

The proof of this construction may be indicated as follows:

Draw GQ and FS parallel to OX, meeting AC respectively in R and T. Since $QI = a$,



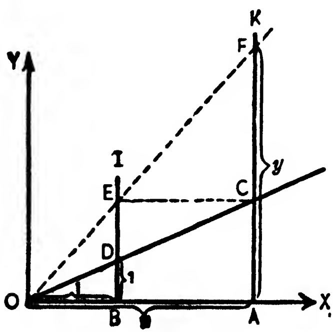
$$RK : a = x : 1, \text{ or } RK = ax.$$

Hence $$SL = b + ax.$$

Since $$TM : SL = x : 1, \quad TM = x(LS),$$

or $$TM = ax^2 + bx, \text{ etc.}$$

For certain fundamental functions the preceding construction becomes very simple. Thus if $y = x^2$, we may construct y (or AF) if x or (OA) is given, as follows:



Make OB and BD equal to unity (different scales may be used for abscissas and ordinates), and let the prolongation of OD meet AK in C. Draw $CE \parallel OX$, then the prolongation of OE meets AK in the re-

quired point F. (This may be proved also by the proportion $OB : BE = OA : AF$ or $1 : x = x : y$.)

The preceding methods for finding $f(x)$ are practical if only a few values of the function have to be found, otherwise the multiplicity of lines will confuse the student.

Another method, which is simpler when many values of the ordinate are required, depends upon the use of certain standard curves, each of which may be used to solve any quadratic, or any cubic, etc. Thus every quadratic equation may be solved by means of the parabola $y = x^2$ and a straight line.* The curve $y = x^2$ is the same, no matter what quadratic we wish to solve, hence it may be printed or mimeographed.

The principle underlying these methods may be illustrated by two examples:

To solve $$ax^2 + bx + c = 0. \quad (1)$$

Let $$\left.\begin{aligned} y &= x^2. \quad (2) \\ ay + bx + c &= 0. \quad (3) \end{aligned}\right\}$$
Then

The solution of (2) and (3) for x produces the required root of (1). But (2) is our standard curve, while (3) is a straight line.

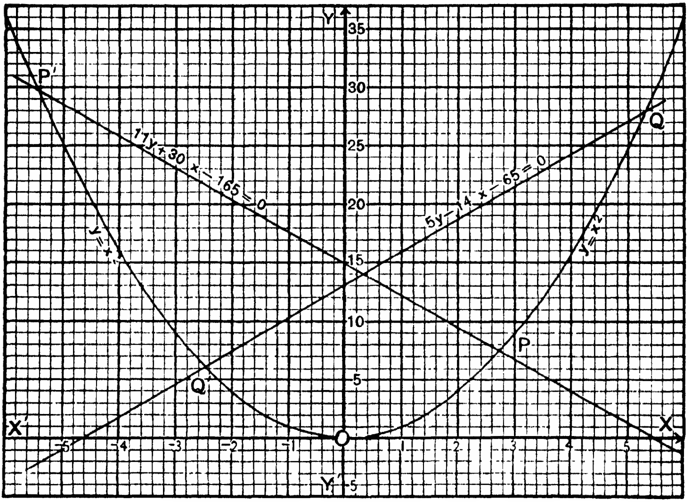
Thus, to solve the equation:

$$11x^2 + 30x - 165 = 0. \quad (1)$$

We let $$y = x^2. \quad (2)$$

Then $$11y + 30x - 165 = 0. \quad (3)$$

* Theoretically a quadratic may be solved by any conic section and a straight line, practical solutions are obtained from the curves: $y = x^2$, $y = \frac{1}{x}$, and $y = x^3$.



In (3), if $x=0$, then $y=15$; if $y=0$, then $x=5\frac{1}{2}$. The straight line joining the points $(0, 15)$ and $(5\frac{1}{2}, 0)$ is the graph of (3), which intersects the graph of (2) in P and P'. By measuring the abscissas of P and P', we have $x=2.7$, or $x=-5.5$.

Similarly, to solve the equation:

$$5x^2-14x-65=0.$$

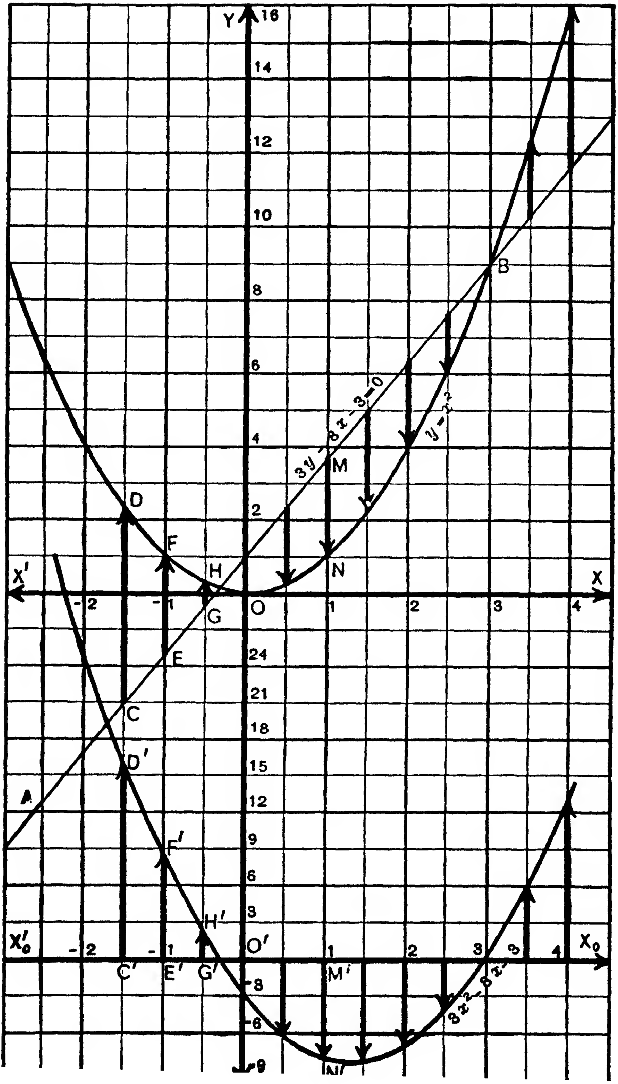
Let $$y=x^2. \quad (1)$$

Then $$5y-14x-65=0. \quad (2)$$

Measuring the abscissas of Q and Q', we obtain

$$x=5.3, \text{ or } x=-2.5.$$

The construction of the graph of ax^2+bx+c by a similar method is shown in the next diagram. Draw the standard curve $y=x^2(DOB)$ and the line $ay+bx+c$. (In the diagram, $a=3$, $b=-8$, $c=-3$.)


Y
X'
X
X'_0
X_0
$3y-8x-3=0$
$y=x^2$
$3x^2-8x-3$
A
B
C
D
E
F
G
H
M
N
O
C'
D'
E'
F'
G'
H'
M'
N'
O'
16
14
12
10
8
6
4
2
24
21
18
15
12
9
6
3
-3
-6
-9
-2
-1
1
2
3
4

Draw a new x-axis X_0X_0' and make the values at the new y-axis a times (3 times) as large as originally. Make $C'D' = CD$, $E'F' = EF$, $G'H' = GH$, $M'N' = MN$, etc., then $D'E'H'N' \cdots$ is the required graph.

In a similar manner, incomplete and complete cubics may be solved, and their graphs be constructed by means of the cubic parabola $y = x^3$ and straight lines, or by $y = x^2$ and circles. It is possible by these methods to solve biquadratics, and to construct their graphs, to find the imaginary roots of quadratics, cubics, biquadratics, etc. For further detail, the reader is referred to the author's Graphic Algebra.

IRRATIONAL AND COMPLEX NUMBERS

What are irrational numbers? — The first numbers ever used in arithmetic were undoubtedly the so-called natural numbers, 1, 2, 3, . . . , and they were then considered as cardinal numbers, *i.e.*, symbols that represent the number of things in a certain group. Using such numbers only, addition and multiplication are operations that are always possible. In order to make division an operation that is always possible, fractions had to be invented, and to accomplish the same for subtraction, negative numbers had to be introduced.†

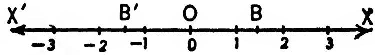
If we wish to make evolution an operation that is

The Macmillan Co., New York, 1908.

† The introduction of negative numbers makes it necessary to adopt the ordinal view of numbers, *i.e.*, the view that numbers are merely marks of order. The use of fractions enables us to use number to indicate the results of measurement, such as the length of a line, the area of a rectangle, etc.

always possible, irrational numbers have to be introduced. Using rational numbers only, $\sqrt{2}$ is an impossible number. For if $\sqrt{2} = \frac{m}{n}$, where m and n have no common factor, then $2 = \frac{m^2}{n^2}$, a result that is evidently impossible. Hence, there was a time when irrational numbers were considered impossible numbers.

The fact that irrational numbers are just as real as rational numbers is made clear by geometric considerations. Every rational number can be represented by a point in the geometric illustration of the number scale, but not every point in that line represents a rational number. Thus if OB equals the hypotenuse of a right triangle whose other sides equal unity, then point B represents an irrational number. Hence, irrational numbers are real numbers.



Definition of irrational numbers. — If we defined an irrational number, as $\sqrt{2}$, by the equation

$$\sqrt{2} \times \sqrt{2} = 2,$$

we would make the mistake of using the symbol $\times$ without defining its meaning. For the symbol of multiplication has thus far been invested with a meaning only if the factors are real, and its meaning when connecting two irrational numbers cannot be defined until irrational numbers are defined.

The definition which is now most widely accepted, but which cannot be fully explained here, is due to the

mathematicians of the Berlin school. It considers an irrational number as a symbol of the division of all *rational* numbers into two classes, each number of one class being distinguished from each number of the other class by a characteristic property.*

It is obviously impossible to teach these matters in a secondary school, and even the method which shows that the $\sqrt{2}$ is the mark of division between two sets of concrete numbers, viz.,

	1	1.4	1.41	1.414	...
and	2	1.5	1.42	1.415	...

and similar discussions have no value, since the student does not understand the reasons that make us search for a definition of irrational which is based upon rational numbers only.

We may, however, point out to him some of the assumptions that have been made in writing:

$$\sqrt{2} \times \sqrt{3} = \sqrt{6}.$$

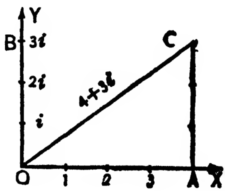
Imaginary numbers. In order to make evolution an operation that is always possible, the introduction of

* We may divide all *rational* numbers into two classes so that all numbers of one class (A) are greater or equal to 2, while all numbers of the other class (B) are less than 2. Then 2 is the mark which divides the two classes. Every number in A is greater than every number in B. In class A there is one number which is smallest, viz., 2, but it is impossible to assign a number in B which is greatest.

Similarly $\sqrt{2}$ divides all rational numbers into two classes A and B. The class A contains all rational numbers whose square is greater than 2, class B contains all rational numbers whose square is less than 2. It is, however, impossible to assign a smallest number in A, or a greatest number in B.

irrational numbers is necessary, but not sufficient. To invest $\sqrt{-1}$ or $\sqrt{-4}$ with a meaning, imaginary numbers must be introduced. Imaginary numbers were considered impossible numbers long after negative and irrational numbers had been accepted. This was mainly due to the fact that a problem has no solution if it leads to a quadratic with imaginary roots. If all algebra were limited to quadratic equations, there would be no practical reason for introducing imaginaries. In solving cubics by Cardan's method, however, the answers appear in an imaginary form if all roots are real (irreducible case). Thus the problem, already mentioned, by Archimedes: "To cut off from a sphere one third of its volume by a plane," leads to a cubic whose roots appear in imaginary form. Hence to obtain the real answer we must be able to operate with imaginaries.

Imaginary numbers are just as real as other numbers, and the view formerly so prevalent that imaginary numbers are impossible is erroneous. The reality of imaginary numbers appears from many topics of higher mathematics. The best elementary illustration is based upon the geometric representation of the number scale. It can easily be shown * that imaginary and complex numbers are represented by points without the line OX.



Thus point B (or line OB) represents $3\,i$, OC represents $4 + 3\,i$.

* See Advanced Algebra, p. 377.

It can be proved that all operations of algebra are always possible, if we use rational, irrational, and complex numbers. Hence no other kind of number has to be introduced into algebra, unless we change the fundamental laws upon which the whole science rests.

The teaching of imaginaries. — Beginners sometimes expect that $(\sqrt{-1})^2$ should equal ± 1, for they claim that: $(\sqrt{-1})^2 = \sqrt{(-1)^2} = \sqrt{1} = \pm 1$. But obviously $(\sqrt{-1})^2$ equals -1 only, since roots are defined by the equation $(\sqrt[n]{a})^n = a$. The erroneous answer $+1$ is introduced by a careless application of the law $(\sqrt[n]{a})^m = \sqrt[n]{a^m}$, which, even in the domain of real number, is true only in regard to the absolute values of the numbers. Thus $(\sqrt{5})^2 = 5$, and only 5, but the above law would give $(\sqrt{5})^2 = \sqrt{25} = \pm 5$, the answer -5 being evidently wrong. Similarly $(\sqrt[4]{1})^8 = 1^2 = +1$, while the above law would produce the wrong answers ± 1, and $\pm i$.,

A similar difficulty occurs in multiplying imaginaries, *e.g.*,

$$\sqrt{-2} \times \sqrt{-3} = i\sqrt{2} \times i\sqrt{3} = i^2\sqrt{6} = -\sqrt{6}.$$

Here the student is inclined to proceed as follows:

$$\sqrt{-3} \times \sqrt{-2} = \sqrt{(-2)(-3)} = \sqrt{6}.$$

The difficulty is caused by the arbitrary restriction of signs. If $\sqrt{-3}$ were taken in its true meaning, viz., $\pm\sqrt{-3}$, and similarly $\sqrt{-2}$ as $\pm\sqrt{-2}$, their product would equal $\pm\sqrt{-6}$. The arbitrary restriction of the

original signs involves the restriction of the answer to $-\sqrt{6}$, and it is worth emphasizing that this result can be obtained only by writing $\sqrt{-a}$ in the form $\sqrt{-1}\sqrt{a}$, or $i\sqrt{a}$. The form $a+bi$ is called the typical form of complex numbers. If the student is aware that all imaginary numbers must be reduced to their typical form before they can be added, subtracted, multiplied, etc., he knows practically all that is needed for operations with complex numbers.

LOGARITHMS

The teaching of the theory. — Since logarithms are introduced into our school curriculum mainly on account of their practical value, it seems to be advisable not to carry study of the theory too far. If thus restricted to the most fundamental propositions, the theory of logarithms becomes very simple, because all propositions are the direct consequences of the definition, which states that $x = \log_b n$, if $b^x = n$. Thus any problem or theorem that may be proposed should be written in the exponential form. For instance, to find log 1, we have to translate the equation $x = \log_b 1$ into the form $b^x = 1$, and the answer is obvious.

To prove that $\log_b(mn) = x + y$, if $x = \log_b m$, and $y = \log_b n$, we have to write hypothesis and conclusion in the exponential form and the proof is evident; viz.,

Hyp.	$x = \log_b m,$	*i.e.*,	$b^x = m.$
	$y = \log_b n,$	*i.e.*,	$b^y = n.$
Con	$x + y = \log_b mn,$	*i.e.*,	$b^{x+y} = m \cdot n.$

Similarly, to express $\log_3 5$ by common logarithm, we write the equation $x = \log_3 5$ in the form $3^x = 5$. This exponential equation solved by the usual method produces the value $x = \frac{\log 5}{\log 3}$.

If still greater simplicity should be required, we may consider the above theorems concretely. Thus,

$$\log 2 = .30103, \quad \textit{i.e.}, \quad 10^{.30103} = 2.$$

$$\log 3 = .47712, \quad \textit{i.e.}, \quad 10^{.47712} = 3.$$

$$\therefore 10^{.77815} = 6,$$

or

$$\log 6 = .77815.$$

Logarithmic calculations. — Although the increasing use of calculating machines has somewhat diminished the practical importance of logarithms, the subject is still the most useful one in elementary algebra. Hence it is necessary to make the student so familiar with the practical use of logarithms that he can do the work accurately and quickly.

We must not, however, carry numerical accuracy to an extreme. Four-place tables are sufficient for many practical purposes, although there are occasions when five-place tables are needed. But tables containing six or more places should not be used in school, since they are needed only in very exceptional cases.

In using any table we ought to take care not to carry the accuracy of the numerical results farther than the table justifies. This does not refer so much to such obvious absurdities as the finding of six places by means of a five-place table, but mainly to using a table as if all

places given were absolutely exact. But limiting a logarithm to five places means neglecting the following places, *i.e.*, it involves an error that may equal $\frac{1}{2}$ a unit of the last place. Adding or subtracting logarithms produces a possible error equal to the sum of all errors. Multiplying a logarithm by six means multiplying the error by six, etc. Thus to find the logarithm of

$$x = \frac{1414 \times 27^5}{.072 \cdot \sqrt[3]{102}},$$

we have the following possible errors.

	Possible Error
log 1414	$\frac{1}{2}$ unit of last place.
5 log 27	$\frac{5}{2}$ unit of last place.
log .072	$\frac{1}{2}$ unit of last place.
$\frac{1}{3}$ log 102	$\frac{1}{6}$ unit of last place.
∴ log x	$3\frac{2}{3}$ unit of last place.

But even if we do not add and subtract at all, the use of the table involves errors, *e.g.*, let

$$\log x = 1.59120.$$

A five-place table gives the answer

$$x = 39.01\tfrac{2}{11} \text{ or } 39.0118.$$

But the numerator (2) is subject to an error of $\frac{1}{2}$, while the denominator, which is the difference of two logarithms, may contain an error of $\frac{2}{2}$. Hence the true value of the fraction $\frac{2}{11}$ may be $\frac{2.5}{10} = .25$, or $\frac{1.5}{12} = .13$.

Consequently the last place of x (8) has no value, and even the preceding place (1) is only an approximation.

If, however, as in the first example, the value of $\log x$ may contain an error of several units of the last place, it is useless to attempt to obtain more than 4 significant figures for the value of x.

Slide Rule. — The slide rule is a very simple instrument which calculates mechanically products, quotients, powers, roots, etc. Its principle is based upon the properties of logarithms, which are most lucidly illustrated by means of this instrument. No teacher who is able to secure some slide rules should neglect to explain to his students the principle and the use of this wonderful little machine.

CHAPTER XXI

THE TEACHING OF TRIGONOMETRY

GENERAL REMARKS

Peculiarities of the subject. — The advantages and disadvantages of the study of trigonometry, as compared with that of other branches of school mathematics, may be briefly summarized as follows:

Advantages.

1. The practical utility of the subject is very great. In fact, of all the branches of secondary school mathematics, trigonometry has by far the largest number of genuine and interesting applications. This applicability is not restricted to school work only. Trigonometry is so widely used in all exact sciences, that it has been called the backbone of applied mathematics. "Not only is it indispensable to the surveyor, to the navigator, to the ophthalmologist, to the mechanical and electrical engineer, but on account of the flexibility of its forms, it is the best school of preparation for the future analyst." (Simon.)

2. The study of trigonometry offers a good field for the training of the students in accuracy and in exactness. In particular it may be used to familiarize the student with a topic which, in spite of its practical importance, is usually neglected, viz., the methods of numerical calculation.

3. The subject is comparatively easy, and contains much that is of interest to young students.

Disadvantages.

1. The disciplinary value of trigonometry is *comparatively* small.

2. Trigonometry requires more memorizing than geometry or algebra.

Courses in trigonometry. — It is evident that the peculiarities which were mentioned in the preceding paragraph must form the guiding principles for the selection of the subject matter for courses in geometry. In particular we should:

1. Emphasize all parts of trigonometry which are applicable to practical problems or which lead indirectly to such applications.

2. Reduce, as far as possible, all topics which require memorizing, or which cannot be applied by the secondary school student.

From the first principle follows the importance of the solution of the right triangle, of the solution of the general triangle, of the calculation of heights and distances, etc. From the second maxim follows that the number of formulæ to be memorized should be reduced to a minimum, and that large portions of certain topics should be omitted. For example, the functions of versed sine and coversed sine should never be mentioned, and the secant and cosecant should be used only rarely. The latter two functions should be omitted entirely, if unfortunately it were not true that so many American textbooks on advanced mathematics make ex-

tensive use of them.* On the continent of Europe the secant and cosecant are almost entirely excluded from school use, tables hardly ever contain them, and textbooks on advanced mathematics use them only sparingly. Some German writers even condemn the use of the cotangent.

If the secant, the cosecant, and the cotangent are studied, it is only necessary to know that these functions are reciprocals of the other three functions. Thus to find $\cot(A+B)$, find first $\tan(A+B)$; to express $\sec\frac{A}{2}$ in terms of $\cos A$, express first $\cos\frac{A}{2}$ in terms of $\cos A$, etc. No explicit formula relating to secant, cosecant, and cotangent should be studied.

Among other topics that may be greatly reduced in volume is the treatment of angles greater than 90°. Angles greater than 360° have practically no value to the secondary school student and even those between 180° and 360° are only very rarely applied. Hence it seems to be absurd to make the student memorize six or more formulæ for the reduction of functions greater than 90° to those of angles less than 90°.

Also the proofs of trigonometric identities and the

* Textbooks often use the secant and cosecant in order to avoid fractions. Thus instead of writing $\sin A = \frac{\sin a}{\sin c}$, some books state $\sin A = \sin a \csc c$. But this not only introduces a function which is not found in the tables, but it also destroys the analogy with plane trigonometry $\left(\sin A = \frac{a}{c}\right)$.

solution of trigonomtric equations is frequently carried farther than the values of these subjects justify.

If there is more time at the disposal of the teacher than is needed for the more practical phase of the work, it would be better invested in studying Moivre's theorem and its connection with the geometric representation of complex numbers, than in proving identities like:

$$\frac{\sin(x+2y)-2\sin(x+y)+\sin x}{\cos(x+2y)-2\cos(x+y)+\cos x}=\tan(x+y).$$

TYPICAL PARTS OF TRIGONOMETRY

Definitions of the trigonometric functions.—The functions are usually defined by one of the following three methods:

1. As quotients of positive and negative coördinates.
2. By means of line values.
3. As quotients of the sides of a right triangle.

The first method is quite general, *i.e.*, it refers to angles in all quadrants, and hence it is frequently given in the opening chapters of textbooks. But, on the other hand, the notion of positive and negative lines, of coördinates, etc., brings into the subject a number of difficulties which may at first be avoided, and hence this method can hardly be recommended as a starting method. The functions of the first quadrant are by far the most important for the high school student, and hence no harm is done in restricting the opening chapters to acute angles.

The method of representing the functions by single

lines is also general, and admirably adapted to an examination of the changes of the functions when the angle changes; but the same objection as was made in the first case makes these definitions unfit for a start.

For the solution of the right triangle, which may be considered the backbone of practical trigonometry, the definitions of the functions as quotients of the sides of a right triangle are fully sufficient, and as this method is exceedingly simple, it seems to be best suited for the beginner. After the student has applied this knowledge to many problems, and has acquired a thorough familiarity with these ratios, general definitions may be introduced. The practice of studying two sets of definitions simultaneously is decidedly unpedagogic.

Solution of the right triangle. — This is probably the most important chapter in the entire elementary trigonometry, and a thorough knowledge of the same will enable the student to solve the greater part of all practical problems that occur in secondary school courses.* It is advisable to take up this subject early in the course.

It is, of course, absurd to subdivide the solution of the right triangle into five cases, since all cases are solved by the same method, viz., the writing and solving of the equation which connects the two given parts and the required part. Thus, if we employ the usual notation, and let $\angle C = 90°$, we would find b in terms of

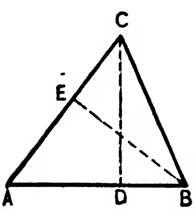
* A student who can devote only 8 or 10 lessons to the study of trigonometry may acquire in this time a fair knowledge of the most practical parts of the subject, if it is largely devoted to the study of the right triangle and its applications.

A and c, by writing the equation which connects b, c, and A, viz.:

$$\cos A = \frac{b}{c}. \quad \therefore b = c \cos A.$$

Such triangles should at first be solved without the use of logarithms, in order to overcome only one difficulty at a time, and to avoid the misconception that trigonometric work is absolutely connected with logarithms. At present, when the use of calculating machines is so general, the solution of triangles without logarithms is more important than formerly.

After the solution of right triangles is thoroughly mastered, this method should be applied to practical problems, as the finding of heights, distances, etc. It should also be applied to the solution of figures, which can be decomposed into right triangles, as the isosceles triangle, the regular polygon, the oblique triangle, etc. Quite complex figures may thus be attacked, if we bear in mind that in these cases a series of right triangles has to be formed such that the solution of each makes possible the solution of the next.



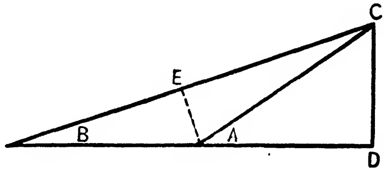
For instance, if in oblique triangle ABC, AB or c, $\angle A$, and $\angle B$ are given, and the altitude CD is required, the first right triangle must necessarily contain the side c and one adjacent angle, *e.g.*, $\angle A$. Hence drop $BE \perp AC$ and solve $\triangle ABE$. Thus we find $BE = c \sin A$, which enables us to solve the adjacent $\triangle ECB$. Since $C = 180 - (A + B)$, it follows that

$$BC = \frac{BE}{\sin C} = \frac{c \sin A}{\sin (180 - A - B)}.$$

Finally we can solve $\triangle CDB$, resulting in the required value,

$$CD = BC \sin B = \frac{c \sin A \sin B}{\sin (A + B)}.$$

Similarly we may solve by means of right triangles the problem of finding the height CD of an object C above two points A and B, if A, B, and C lie in a vertical plane and angles A and B and distance AB are known.



The first right triangle must necessarily contain AB and $\angle A$. Hence draw $AE \perp BC$, and find $AE = AB \sin B$. Since $\angle ECA = \angle A - \angle B$, we are now able to solve $\triangle AEC$, and we find

$$AC = \frac{AE}{\sin (A - B)} = \frac{AB \sin B}{\sin (A - B)}.$$

Finally $\triangle ACD$ produces

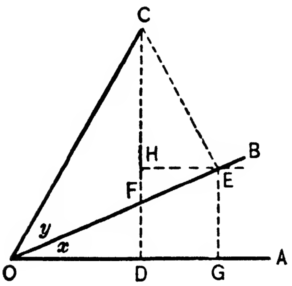
$$CD = AC \sin B = \frac{AB \sin A \sin B}{(\sin A - B)}.$$

In problems of the preceding kind it is advisable to determine the final answer at first in algebraic form, and if numerical values are required to substitute numbers in the general answer. The substitution of numbers at the very beginning is not only awkward, since the writing of $20° \ 4' \ 16''$ is more complex than the writing of B, but it frequently involves unnecessary calculations. Thus, in the preceding problem the finding of the numerical values of AE and AC is unnecessary.

Another illustration of the method of solving complex figures by means of a chain of right triangles is the analysis of the addition theorem:

$$\sin (x + y) = \sin x \cos y + \cos x \sin y.$$

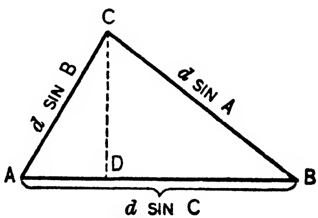
To find sin $(x+y)$ we may represent this quantity by a line, *i.e.*, we may make $OC = 1$, and draw $CD \perp OA$, then CD represents the unknown line, while OC is the only given line. In regard to the angles we must bear in mind that we are dealing not with given angles themselves, but with angles whose functions are given.



The first triangle must contain OC and the only adjacent angle whose functions are known, viz., y. Hence, drop $CE \perp OB$. The right triangle OCE produces the values $CE = \sin y$, and $OE = \cos y$. Before proceeding, we have to survey the diagram and determine all angles whose functions are known (compare Chapter XI). Such angles are $CFB = OFD = 90 - x$, and $\angle FCE = 90° - \angle CFB = x$.

To construct a right triangle which contains CE, we must associate this line with $\angle FCE$, which is the only adjacent angle. Hence, draw $EH \perp CD$ and find the values $CH = \cos x \sin y$ and $EH = \sin x \cos y$. Similarly the line OE must be associated with $\angle x$, *i.e.*, we drop $EG \perp OA$, and determine the values of OG and EG. Obviously the required line CD equals $CH + EG$, which leads to the required equation.*

* A very short proof of the addition theorem may be based upon the fact that the sides of $\Delta\ ABC$ may be represented in the form $BC = d \sin A$, $CA = d \sin B$, and $AB = d \sin C = d \sin (A + B)$.



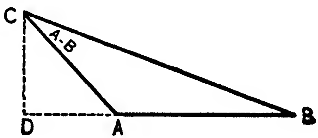
Drawing $CD \perp AB$, we have:

$$AB = BD + DA,$$

or $d \sin (A + B) = d \sin A \cos B + d \cos A \sin B$.

$\therefore \sin (A + B) = \sin A \cos B + \cos A \sin B$.

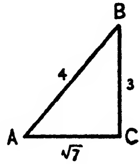
In a similar manner the next diagram produces $\sin (A - B)$.



Expressing one function in terms of another. — While this is frequently done by means of the fundamental formulæ which connect the functions, a more practical way is based upon the right triangle, as illustrated in the following examples. It is true that the results thus obtained relate directly only to angles less than 90°, but the proper modification of the sign makes the results applicable to all quadrants.

Ex. 1. Given $\sin A = \frac{3}{4}$, required all other functions of A.

If in right triangle ABC, we make $BC = 3$, and $AB = 4$, then A equals the given angle. Obviously $AC = \sqrt{7}$. Therefore, all functions can be determined by inspection of the diagram.



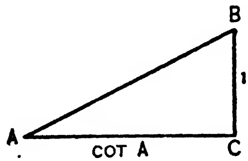
Ex. 2. Given $\tan A = 2$, required all other functions of A.

Here we make $BC = 2$, and $AC = 1$. Then $AB = \sqrt{5}$, and all functions can be read from the diagram.

Ex. 3. Given $\sec A = \frac{m}{n}$, find the other functions of A.

Here $AB = m$, $AC = n$, and hence $BC = \sqrt{m^2 - n^2}$.

Ex. 4. Given $\csc A = m$, required $\cos A$. Make $AB = m$, $BC = 1$. Hence $AC = \sqrt{m^2 - 1}$, and $\cos A = \frac{\sqrt{m^2 - 1}}{m}$.



Ex. 5. Express all functions in terms of $\cot A$.

Let $AC = \cot A$ and $BC = 1$. Then $AB = \sqrt{\cot^2 A + 1}$, and all other functions may be determined by inspection.

Methods for proving identities. — The simplest method for demonstrating trigonometric identities involving only one angle reduces them to identities involving the three sides of a right triangle, which latter identities are proved by the usual algebraic method. This method is often lengthy, but quite easy for the beginner.

Thus to prove that

$$\frac{\sin A}{1 + \cos A} + \frac{1 + \cos A}{\sin A} = 2 \csc A,$$

we express the functions involved as quotients of lines, *i.e.*:

$$\frac{\frac{a}{c}}{1 + \frac{b}{c}} + \frac{1 + \frac{b}{c}}{\frac{a}{c}} = 2\frac{c}{a}. \quad \text{This is true if}$$

$$\frac{a}{c + b} + \frac{c + b}{a} = \frac{2c}{a}. \quad \text{This is true if}$$

$$a^2 + c^2 + 2cb + b^2 = 2c^2 + 2cb. \quad \text{This is true if}$$

$$a^2 + b^2 = c^2.$$

But the last equation is true, hence the given identity is proved.

All fundamental formulæ which connect trigonometric functions, as $\sin^2 A + \cos^2 A = 1$, $\dfrac{\sin A}{\cos A} = \tan A$, etc., may be proved by this method.

The preceding method may be simplified by making one of the sides of the right triangle equal to unity, usually that one which occurs frequently in the denominator.

Thus in the preceding example we would make $c = 1$. Then the identity would mean

$$\frac{a}{1 + b} + \frac{1 + a}{a} = \frac{2}{a},$$

which easily reduces to

$$a^2 + b^2 = 1.$$

A method which usually leads to much shorter proofs, but which requires a little more skill, expresses all functions involved in terms of two functions (usually sine and cosine). If this does not lead to a demonstration, all functions are expressed in terms of one function.

In the above example, this method produces:

$$\frac{\sin A}{1+\cos A}+\frac{1+\cos A}{\sin A}=\frac{2}{\sin A},$$

or
$$\sin^2 A+(1+\cos A)^2=2(1+\cos A),$$

or
$$\sin^2 A+\cos^2 A=1.$$

If the functions involved are not all functions of the same angle A, but some are functions of $\frac{A}{2}$, or $2A$, or $3A$, etc., all functions have at first to be reduced to functions of one angle.

Thus to prove

$$2\sin x+\sin 2x=\frac{2\sin^3 x}{1-\cos x},$$

$\sin 2x$ must be expressed in terms of functions of x, *i.e.*, we must substitute $2\sin x\cos x$ for $\sin 2x$, thereby reducing the problem to one of the preceding kind.

Similarly to prove that

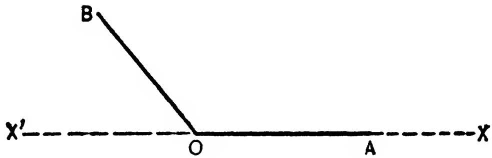
$$\tan\frac{x}{2}+\cot\frac{x}{2}=2\csc x,$$

we must either express $\csc x$ in terms of $\frac{x}{2}$, or $\tan\frac{x}{2}$ and $\cot\frac{x}{2}$ in terms of x.

Functions of angles greater than 90°. — To express the functions of angles greater than 90° in terms of

functions of angles less than 90°, usually a large number of formulæ are memorized.

If we remember that an angle AOB is usually supposed to be generated by a counter-clockwise rotation, and that hence OB is the terminal radius, and if we

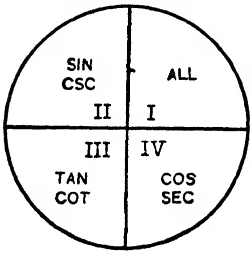


denote XX', *i.e.*, the produced initial radius, as x-axis, we may reduce all functions by the following theorem:

A function of any angle = ± the same function of the acute angle formed by terminal radius and x-axis.

The sign ± here does not indicate that both signs are true, but that either + or − has to be selected. The selection of the sign is made according to the annexed diagram, which gives the positive functions for each quadrant. All other functions are negative. Since the positive functions of each quadrant are reciprocals of each other, it is only necessary to remember that the following functions are positive: in quadrant II, the sine; in quadrant III, the tangent; and in quadrant IV, the cosine.*

POSITIVE FUNCTIONS



* Even these three parts need not be memorized if the student has a clear conception of the representation of functions by single lines.

Thus to reduce $\cos 245°$, we consider that an angle of $245°$ lies in quadrant III, and that hence the cosine is negative. Since the acute angle formed by terminal radius and x-axis is $65°$, it follows that $\cos 245° = - \cos 65°$.

Inverse trigonometric functions. Inverse trigonometric functions are of comparatively small importance to the students of secondary schools. The understanding of the meaning and the use of symbols like $\sin^{-1} m$, $\tan^{-1} n$,* etc., is increased by reading them always as "angles." Thus read

$\sin^{-1} \frac{3}{4}$ as "the angle whose sine $= \frac{3}{4}$."

$\tan^{-1} x$ as "the angle whose tangent $= x$."

This will enable the student to solve many of the simpler problems without any special method, *e.g.*, the finding of the following expressions: $\sin^{-1} \frac{1}{2}$, $\cos^{-1} \frac{1}{2}\sqrt{3}$, $\sin(\tan^{-1} 1)$, $\sec(\tan^{-1} n)$, etc.

For more complex cases, it is advisable to introduce a symbol for the angle whose function is represented. Thus to find

$$\tan(2 \tan^{-1} n), \tag{1}$$

let

$$\tan^{-1} n = A, \tag{2}$$

then

$$\tan A = n.$$

Therefore

$$\tan(2 \tan^{-1} n) = \tan 2A = \frac{2 \tan A}{1 - \tan^2 A} = \frac{2n}{1 - n^2}.$$

* The symbol $\sin^{-1} m$ is rather unfortunate, since it may mean $(\sin m)^{-1}$.

Line (2) is typical for the substitutions that have to be made in such examples.

To illustrate examples involving two angles, let us find

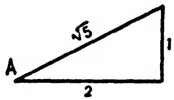
$$\tan\left(\sin^{-1}\frac{1}{\sqrt{5}}+\tan^{-1}\frac{1}{3}\right).$$

Let $\sin^{-1}\dfrac{1}{\sqrt{5}}=A$, then $\sin A=\dfrac{1}{\sqrt{5}}$,

and $\tan^{-1}\dfrac{1}{3}=B$, then $\tan B=\dfrac{1}{3}$.

$$\tan\left(\sin^{-1}\frac{1}{\sqrt{5}}+\tan^{-1}\frac{1}{3}\right)=\tan(A+B)$$

$$=\frac{\tan A+\tan B}{1-\tan A\tan B}.$$



But since $\qquad\sin A=\dfrac{1}{\sqrt{5}},$

$$\tan A=\tfrac{1}{2},$$

hence $\qquad\tan(A+B)=\dfrac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}=1.$

If a theorem has to be proved, simplify one or both members. Thus to prove that $\sin(\sin^{-1}x+\sin^{-1}y)=x\sqrt{1-y^2}+y\sqrt{1-x^2}$ simplify $\sin(\sin^{-1}x+\sin^{-1}y)$ as above.

If both members of an identity are inverse trigono-

metric functions, take the tangent (or other function) of both members. Thus to prove

$$\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5} = \frac{\pi}{4} + \tan^{-1}\frac{8}{19}$$

take the tangent of both members, or prove that

$$\tan\left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5}\right) = \tan\left(\frac{\pi}{4} + \tan^{-1}\frac{8}{19}\right).$$

Both members simplified produce the result $\frac{27}{11}$. Hence the identity is true.

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